3
Functions

FUNCTION *Let X and Y be two nonempty sets. Then, a relation f from X to Y is called a function, if every element in X has a unique image in Y, and we write,* $f : X \to Y$.

Thus, a relation f from X to Y is a function, if dom $(f) = X$ and no two distinct ordered pairs in f have the same first coordinate.

If $(x, y) \in f$, we write, $f(x) = y$.

Here, y is called the *image of* x *under* f and x is called the *pre-image* of y.

If $f : X \to Y$ then *dom* $(f) = X$ and *range* $(f) \subseteq Y$.

Also, Y is called the *co-domain* of f.

IMAGE OF A SUBSET

Let $f : X \to Y$ and let $A \subset X$.

Then, the image of A under f is defined as

$$f(A) = \{f(x) : x \in A\}.$$

Clearly, $f(A) \subseteq Y$.

EXAMPLE 1 *Let* $X = \{1, 2, 3, 4\}$ *and* $Y = \{1, 4, 9, 16, 25\}$.

Let $f = \{(x, y) : x \in X, y \in Y$ *and* $y = x^2\}$.

(i) Show that f is a function from X to Y. Find its domain and range.

(ii) Draw a pictorial representation of the above function.

(iii) If $A = \{2, 3, 4\}$, *find* $f(A)$.

SOLUTION (i) We have, $f = \{(x, y) : x \in X, y \in Y$ and $y = x^2\}$.

Giving different values to x from the set X and getting the corresponding value of $y = x^2$, we get

$$f = \{(1, 1), (2, 4), (3, 9), (4, 16)\}.$$

Clearly, every element in X has a unique image in Y.

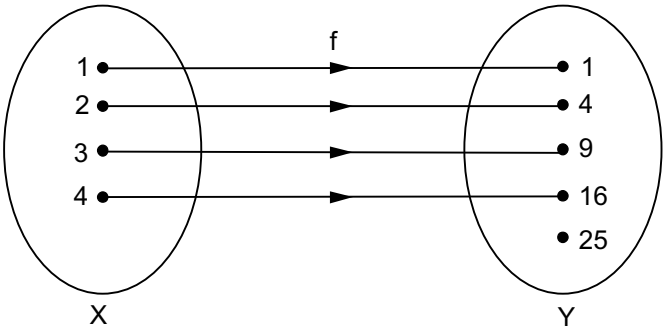
Hence, f is a function from X to Y.

Dom $(f) = \{1, 2, 3, 4\} = X$.

Range $(f) = \{1, 4, 9, 16\} \subset Y$.

Clearly, $25 \in Y$ does not have its pre-image in X.

(ii) A pictorial representation of the above mapping f is given below.



(iii) Now, let $A = \{2, 3, 4\}$. Then,

$$f(2) = 2^2 = 4, \quad f(3) = 3^2 = 9 \text{ and } f(4) = 4^2 = 16.$$

$$\therefore \quad f(A) = \{f(x) : x \in A\} = \{4, 9, 16\}.$$

RELATION AS A PARTICULAR CASE OF A FUNCTION

A relation f from X to Y is a function if dom $(f) = X$ and no two distinct ordered pairs in f have the same first coordinate.

If $(x, y) \in f$, we write, $f(x) = y$.

EXAMPLE 2 *Let* $X = \{2, 3, 4, 5\}$ *and* $Y = \{7, 9, 11, 13, 15, 17\}$.

Define a relation f from X to Y by:

$$f = \{(x, y) : x \in X, y \in Y \text{ and } y = 2x + 3\}.$$

(i) Write f in roster form.

(ii) Find dom (f) and range (f).

(iii) Show that f is a function from X to Y.

SOLUTION Here $X = \{2, 3, 4, 5\}$ and $Y = 2x + 3$.

Now, $x = 2 \Rightarrow y = (2 \times 2 + 3) = 7$,

$x = 3 \Rightarrow y = (2 \times 3 + 3) = 9$,

$x = 4 \Rightarrow y = (2 \times 4 + 3) = 11$,

$x = 5 \Rightarrow y = (2 \times 5 + 3) = 13$.

(i) $\therefore \quad f = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$.

(ii) Clearly, dom $(f) = \{2, 3, 4, 5\}$ and range $(f) = \{7, 9, 11, 13\} \subset Y$.

(iii) It is clear that no two distinct ordered pairs in f have the same first coordinate.

$\therefore \quad f$ is a function from X to Y.

EXAMPLE 3 *Which of the following relations are functions? Give reasons. In case of a function, find its domain and range.*

(i) $f = \{(1, 3), (1, 5), (2, 3), (2, 5)\}$

(ii) $g = \{(2, 1), (5, 1), (8, 1), (11, 1)\}$

(iii) $h = \{(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6)\}$

SOLUTION (i) $f = \{(1, 3), (1, 5), (2, 3), (2, 5)\}$.

Here, one element, namely 1 has two images 3 and 5 under f.

$\therefore \quad f$ is not a function.

(ii) $g = \{(2, 1), (5, 1), (8, 1), (11, 1)\}$.

Clearly, no two distinct ordered pairs in g have the same first coordinate. So, g is a function.

$\therefore$ dom $(g) = \{2, 5, 8, 11\}$ and range $g = \{1\}$.

(iii) $h = \{(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6)\}$.

Clearly, no two distinct ordered pairs in h have the same first coordinate. So, h is a function.

$\therefore$ dom $(h) = \{2, 4, 6, 8, 10, 12\}$ and range $(h) = \{1, 2, 3, 4, 5, 6\}$.

EXAMPLE 4 *Let* $A = \{-2, -1, 0, 1, 2\}$ *and* $f : A \to Z : f(x) = x^2 - 2x - 3$.

Find (i) range (f) (ii) pre-images of 6, –3 *and* 5.

SOLUTION We have

$f(-2) = (-2)^2 - 2 \times (-2) - 3 = (4 + 4 - 3) = 5;$

$f(-1) = (-1)^2 - 2 \times (-1) - 3 = 0;$

$f(0) = -3;$

$f(1) = (1 - 2 - 3) = -4$ and $f(2) = (2^2 - 2 \times 2 - 3) = -3.$

$\therefore$ $f = \{(-2, 5), (-1, 0), (0, -3), (1, -4), (2, -3)\}$.

(i) Range $(f) = \{5, 0, -3, -4\}$.

(ii) $f(x) = 6 \Rightarrow x^2 - 2x - 3 = 6$

$$\Rightarrow x^2 - 2x - 9 = 0 \Rightarrow x = \frac{2 \pm \sqrt{4 + 36}}{2} = (1 \pm \sqrt{10}) \notin A.$$

$\therefore$ 6 has no pre-image in A.

$f(x) = -3 \Rightarrow x^2 - 2x - 3 = -3$

$\Rightarrow x^2 - 2x = 0 \Rightarrow x(x - 2) = 0 \Rightarrow x = 0$ or $x = 2.$

Clearly, $0, 2 \in A$.

So, the pre-images of –3 are 0 and 2.

$f(x) = 5 \Rightarrow x^2 - 2x - 3 = 5$

$\Rightarrow x^2 - 2x - 8 = 0 \Rightarrow x^2 - 4x + 2x - 8 = 0$

$\Rightarrow x(x - 4) + 2(x - 4) = 0 \Rightarrow (x - 4)(x + 2) = 0$

$\Rightarrow x = -2$ or $x = 4.$

Now, $-2 \in A$ but $4 \notin A$.

So, the pre-image of 5 is –2.

EQUAL FUNCTIONS

Two functions f and g are said to be equal if

(i) $dom\,(f) = dom\,(g)$ *(ii) co-domain of* f *= co-domain of g*

(iii) $f(x) = g(x)$ *for every x in their common domain.*

EXAMPLE 5 *Let* $A = \{1, 2\}$ *and* $B = \{3, 6\}$ *and f and g be functions from A to B, defined by* $f(x) = 3x$ *and* $g(x) = x^2 + 2$. *Show that* $f = g$.

SOLUTION Clearly, we have

$$\text{dom}(f) = \text{dom}(g) = \{1, 2\}.$$

Co-domain of f = co-domain of $g = \{3, 6\}$.

Also, $f(1) = (3 \times 1) = 3$, $f(2) = (3 \times 2) = 6$;

$$g(1) = (1^2 + 2) = 3, \quad g(2) = (2^2 + 2) = 6.$$

$\therefore$ $f(1) = g(1)$ and $f(2) = g(2)$.

Thus, $f(x) = g(x)$ for all $x \in A$.

Hence, $f = g$.

EXAMPLE 6 *Let* $f : Z \to Z : f(x) = x^2$ *and* $g : Z \to Z : g(x) = |x|^2$ *for all* $x \in Z$. *Show that* $f = g$.

SOLUTION We have

$$\text{dom}(f) = \text{dom}(g) = Z$$

and co-domain of f = co-domain of $g = Z$.

Also, for all $x \in Z$, we have

$$f(x) = x^2 \text{ and } g(x) = |x|^2 = x^2.$$

$\therefore$ $f(x) = g(x)$ for all $x \in Z$.

Hence, $f = g$.

EXAMPLE 7 *Let* $f : R \to R : f(x) = x + 2$ *and* $g : R - \{2\} \to R : g(x) = \dfrac{x^2 - 4}{x - 2}$.

Show that $f \neq g$. *Re-define f and g such that* $f = g$.

SOLUTION We have, $\text{dom}(f) = R$ and $\text{dom}(g) = R - \{2\}$.

Since $\text{dom}(f) \neq \text{dom}(g)$, so we have $f \neq g$.

For every real number $x \neq 2$, we have

$$g(x) = \frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{(x - 2)} = (x + 2) \qquad [\because (x - 2) \neq 0].$$

Thus, $f(x) = g(x)$ for all $x \in R - \{2\}$.

$\therefore$ $f = g$ only when, they are re-defined as under:

$$f : R - \{2\} \to R : f(x) = x + 2 \text{ and } g : R - \{2\} \to R : g(x) = \frac{x^2 - 4}{x - 2}.$$

EXAMPLE 8 *Let* $f = \{(-1, -3), (0, -1), (1, 1), (2, 3)\}$ *be a function, described by the formula,* $f(x) = \alpha x + \beta$. *Then, find the values of* α *and* β. *Also, find the formula.*

SOLUTION Here $f(x) = \alpha x + \beta$. … (i)

Also, $f(-1) = -3$, $f(0) = -1$, $f(1) = 1$ and $f(2) = 3$. [given]

Putting $x = -1$ and $f(-1) = -3$ in (i), we get

$$-\alpha + \beta = -3 \Rightarrow \alpha - \beta = 3. \qquad \text{… (ii)}$$

Putting $x = 0$ and $f(0) = -1$ in (i), we get

$$\alpha \times 0 + \beta = -1 \Rightarrow \beta = -1. \qquad \text{… (iii)}$$

Putting $\beta = -1$ from (iii) in (ii), we get $\alpha = 2$.

$\therefore$ $\alpha = 2$ and $\beta = -1$.

Hence, $f(x) = 2x - 1$ is the required formula.

EXAMPLE 9 *Let* $f : R \to R : f(x) = x^2 + 3$.

Find the pre-images of each of the following under f:

(i) 19 *(ii)* 28 *(iii)* 2

SOLUTION Given: $f(x) = x^2 + 3$.

(i) Let *x* be the pre-image of 19. Then,

$$f(x) = 19 \Rightarrow x^2 + 3 = 19 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4.$$

$\therefore$ 4 and –4 are the pre-images of 19.

(ii) Let *x* be the pre-image of 28. Then,

$$f(x) = 28 \Rightarrow x^2 + 3 = 28 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5.$$

$\therefore$ 5 and –5 are the pre-images of 28.

(iii) Let *x* be the pre-image of 2. Then,

$$f(x) = 2 \Rightarrow x^2 + 3 = 2 \Rightarrow x^2 = -1.$$

But, no real value of *x* satisfies the equation, $x^2 = -1$.

$\therefore$ 2 does not have any pre-image under *f*.

INVERSE OF A FUNCTION

Let $f : X \to Y$ and let $y \in Y$.

Then, we define, $f^{-1}(y) = \{x \in X : f(x) = y\}$ = *set of pre-images of y.*

f^{-1} *is called the inverse of f.*

REMARK In above example 9, we have

(i) $f^{-1}\{19\} = \{-4, 4\}$ (ii) $f^{-1}\{28\} = \{-5, 5\}$ (iii) $f^{-1}\{2\} = \phi$.

EXAMPLE 10 *Let* $f : R \to R : f(x) = x^2 + 1$.

Find (i) $f^{-1}\{-4\}$ *(ii)* $f^{-1}\{10\}$ *(iii)* $f^{-1}\{5, 17\}$.

SOLUTION It is given that $f(x) = x^2 + 1$.

(i) Let $f^{-1}(-4) = x$. Then,

$$f(x) = -4 \Rightarrow x^2 + 1 = -4 \Rightarrow x^2 = -5.$$

But, there is no real value of *x* whose square is –5.

$\therefore$ $f^{-1}\{-4\} = \phi$.

(ii) Let $f^{-1}(10) = x$. Then,

$$f(x) = 10 \Rightarrow x^2 + 1 = 10 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3.$$

$\therefore$ $f^{-1}\{10\} = \{-3, 3\}$.

(iii) Let $f^{-1}(5) = x$. Then,

$$f(x) = 5 \Rightarrow x^2 + 1 = 5 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2.$$

$\therefore$ $f^{-1}\{5\} = \{-2, 2\}$.

Let $f^{-1}(17) = x$. Then,

$$f(x) = 17 \Rightarrow x^2 + 1 = 17 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4.$$

$\therefore$ $f^{-1}\{17\} = \{-4, 4\}$.

Hence, $f^{-1}\{5, 17\} = \{-2, 2, -4, 4\}$.

EXAMPLE 11 *Let* $f : R \to R$, *defined by*

$$f(x) = \begin{cases} 1, \text{ if } x \in Q \\ -1, \text{ if } x \notin Q. \end{cases}$$

Find (i) $f\left(\frac{1}{2}\right)$ *(ii)* $f(0.34)$ *(iii)* $f(\sqrt{2})$ *(iv)* $f(\pi)$

(v) range(f) *(vi)* $f^{-1}\{1\}$ *(vii)* $f^{-1}\{-1\}$

SOLUTION Since each one of $\frac{1}{2}$ and 0.34 is rational, we have

(i) $f\left(\frac{1}{2}\right) = 1$ and (ii) $f(0.34) = 1$.

Since each one of $\sqrt{2}$ and π is irrational, we have

(iii) $f(\sqrt{2}) = -1$ and (iv) $f(\pi) = -1$.

(v) range $(f) = \{f(x) : x \in R\}$

$= \{f(x) : x \in Q\} \cup \{f(x) : x \in R - Q\} = \{1, -1\}$.

(vi) $f^{-1}\{1\} = \{x : f(x) = 1\} = Q$.

(vii) $f^{-1}\{-1\} = \{x : f(x) = -1\} = (R - Q)$.

EXAMPLE 12 *Let* $f : R \to R$, *defined by*

$$f(x) = \begin{cases} 3x - 2, & x < 0 \\ 1, & x = 0 \\ 4x + 1, & x > 0. \end{cases}$$

Find (i) $f(2)$ *(ii)* $f(-2)$ *(iii)* $f(0)$ *(iv)* $f(3.5)$.

SOLUTION Clearly, we have

(i) $f(2) = (4 \times 2 + 1) = 9$ $[\because\ 2 > 0]$.

(ii) $f(-2) = \{3 \times (-2) - 2\} = -8$ $[\because\ -2 < 0]$.

(iii) $f(0) = 1$.

(iv) $f(3.5) = (4 \times 3.5 + 1) = 15$ $[\because\ 3.5 > 0]$.

EXAMPLE 13 *Consider the relation*

$$f(x) = \begin{cases} x^2, 0 \le x \le 2 \\ 3x, 2 \le x \le 10. \end{cases}$$

Show that f is not a function.

SOLUTION We have, dom $(f) = [0, 10] \subset R$.

Now, $f(2) = 2^2 = 4$ and also, $f(2) = (3 \times 2) = 6$.

As such, a single element, namely 2 has two distinct images, namely 4 and 6.

$\therefore$ f is not a function.

EXAMPLE 14 *Consider the relation*

$$g(x) = \begin{cases} x^2, 0 \le x \le 3 \\ 3x, 3 \le x \le 10. \end{cases}$$

Show that g is a function.

SOLUTION We have, dom $(g) = [0, 10] \subset R$.

It is easy to verify that every element in [0, 10] is associated with a unique real number under *g*.

In particular $g(3) = 3^2 = 9$ and also $g(3) = 3 \times 3 = 9$, i.e., 3 has a unique image, namely 9.

$\therefore$ *g* is a function from [0, 10] into *R*.

EXERCISE 3A

1. Define a function as a set of ordered pairs.
2. Define a function as a correspondence between two sets.
3. What is the fundamental difference between a relation and a function? Is every relation a function?
4. Let $X = \{1, 2, 3, 4\}$, $Y = \{1, 5, 9, 11, 15, 16\}$ and $f = \{(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)\}$.
   Are the following true?
   (i) *f* is a relation from *X* to *Y* (ii) *f* is a function from *X* to *Y*. Justify your answer in each case.
5. Let $X = \{-1, 0, 3, 7, 9\}$ and $f : X \to R : f(x) = x^3 + 1$.
   Express the function *f* as set of ordered pairs.
6. Let $A = \{-1, 0, 1, 2\}$ and $B = \{2, 3, 4, 5\}$. Find which of the following are functions from *A* to *B*. Give reason.
   (i) $f = \{(-1, 2), (-1, 3), (0, 4), (1, 5)\}$
   (ii) $g = \{(0, 2), (1, 3), (2, 4)\}$
   (iii) $h = \{(-1, 2), (0, 3), (1, 4), (2, 5)\}$
7. Let $A = \{1, 2\}$ and $B = \{2, 4, 6\}$. Let $f = \{(x, y) : x \in A, y \in B$ and $y > 2x + 1\}$.
   Write *f* as a set of ordered pairs.
   Show that *f* is a relation but not a function from *A* to *B*.
8. Let $A = \{0, 1, 2\}$ and $B = \{3, 5, 7, 9\}$. Let $f = \{(x, y) : x \in A, y \in B$ and $y = 2x + 3\}$.
   Write *f* as a set of ordered pairs.
   Show that *f* is a function from *A* to *B*.
   Find dom (f) and range (f).
9. Let $A = \{2, 3, 5, 7\}$ and $B = \{3, 5, 9, 13, 15\}$.
   Let $f = \{(x, y) : x \in A, y \in B$ and $y = 2x - 1\}$.
   Write *f* in roster form. Show that *f* is a function from *A* to *B*.
   Find the domain and range of *f*.
10. Let $g = \{(1, 2), (2, 5), (3, 8), (4, 10), (5, 12), (6, 12)\}$.
    Is *g* a function? If yes, find its domain and range.
    If no, give reason.
11. Let $f = \{(0, -5), (1, -2), (2, 1), (3, 4), (4, 7)\}$ be a linear function from *Z* into *Z*.
    Write an expression for *f*.

    Hint Let $f(x) = ax + b$.

12. If $f(x) = x^3$, find the value of $\dfrac{\{f(5) - f(1)\}}{(5-1)}$.

13. If $f(x) = x^2$, find the value of $\dfrac{\{f(1.1) - f(1)\}}{(1.1) - 1}$.

14. Let $A = \{12, 13, 14, 15, 16, 17\}$ and $f : A \to Z : f(x)$ = highest prime factor of x. Find range (f).

15. Let R^+ be the set of all positive real numbers.
Let $f : R^+ \to R : f(x) = \log_e x$.
Find (i) range (f) (ii) $\{x : x \in R^+ \text{ and } f(x) = -2\}$.
(iii) Find out whether $f(xy) = f(x) + f(y)$ for all $x, y \in R^+$.

16. Let $f : R \to R : f(x) = 2^x$.
Find (i) range (f) (ii) $\{x : f(x) = 1\}$.
(iii) Find out whether $f(x + y) = f(x) \cdot f(y)$ for all $x, y \in R$.

17. Let $f : R \to R : f(x) = x^2$ and $g : C \to C : g(x) = x^2$, where C is the set of all complex numbers.
Show that $f \neq g$.

18. f, g and h are three functions defined from R to R as follows:
(i) $f(x) = x^2$ (ii) $g(x) = x^2 + 1$ (iii) $h(x) = \sin x$
Then, find the range of each function.

19. Let $f : R \to R : f(x) = x^2 + 1$. Find (i) $f^{-1}\{10\}$ (ii) $f^{-1}\{-3\}$.

20. The function $F(x) = \dfrac{9x}{5} + 32$ is the formula to convert x°C to Fahrenheit units. Find
(i) $F(0)$, (ii) $F(-10)$, (iii) the value of x when $f(x) = 212$.
Interpret the result in each case.

Hints (i) $f(0) = \left\{\dfrac{9 \times 0}{5} + 32\right\} = 32 \Rightarrow 0°\text{C} = 32°\text{F}$.

(ii) $F(-10) = \left\{\dfrac{9 \times (-10)}{5} + 32\right\} = 14 \Rightarrow (-10)°\text{C} = 14°\text{F}$.

(iii) $F(x) = 212 \Rightarrow \dfrac{9x}{5} + 32 = 212 \Rightarrow x = 100 \Rightarrow 212°\text{F} = 100°\text{C}$.

ANSWERS (EXERCISE 3A)

4. (i) Yes (ii) No **5.** $f = \{(-1, 0), (0, 1), (3, 28), (7, 344), (9, 730)\}$

6. (i) No, since one element, namely –1, has two different images.
(ii) No, since dom $(g) \neq A$.
(iii) Yes, since each element in A has a unique image in B.

7. $f = \{(1, 4), (1, 6), (2, 6)\}$

8. $f = \{(0, 3), (1, 5), (2, 7)\}$, dom $(f) = \{0, 1, 2\}$, range $(f) = \{3, 5, 7\}$

9. $f = \{(2, 3), (3, 5), (5, 9), (7, 13)\}$, dom $(f) = \{2, 3, 5, 7\}$, range $(f) = \{3, 5, 9, 13\}$

10. Yes, dom $(f) = \{1, 2, 3, 4, 5, 6\}$ and range $(f) = \{2, 5, 8, 10, 12\}$

11. $f(x) = 3x - 5$ **12.** 31 **13.** 2.1 **14.** range $(f) = \{3, 13, 7, 5, 2, 17\}$

15. (i) R (ii) $\{e^{-2}\}$ (iii) Yes **16.** (i) R^+ (ii) $\{0\}$ (iii) Yes

18. (i) $R^+ = \{x \in R : x \geq 0\}$ (ii) $\{x \in R : x \geq 1\}$ (iii) $\{x \in R : -1 \leq x \leq 1\}$

19. (i) $\{-3, 3\}$ (ii) ϕ **20.** (i) 32 (ii) 14 (iii) $x = 100$

HINTS TO SOME SELECTED QUESTIONS

4. (i) Since $f \subset X \times Y$, so f is a relation from X to Y.

(ii) Dom $(f) = X$ and range $(f) = \{1, 5, 9, 11\} \subset Y$.

But, two different ordered pairs namely (2, 9) and (2, 11) have the same first coordinate. So, f is not a function.

6. (i) No, since two different ordered pairs (–1, 2) and (–1, 3) have the same first coordinate.

(ii) No, since dom $(g) = \{0, 1, 2\} \neq A$.

(iii) Yes, dom $(h) = A$ and no two different ordered pairs have the same first coordinate.

11. $f(0) = -5 \Rightarrow a \times 0 + b = -5 \Rightarrow b = -5.$

$f(1) = -2 \Rightarrow a \times 1 + b = -2 \Rightarrow a - 5 = -2 \Rightarrow a = 3.$

12. $\dfrac{\{f(5) - f(1)\}}{(5-1)} = \dfrac{\{(5)^3 - 1^3\}}{4} = \dfrac{(125-1)}{4} = \dfrac{124}{4} = 31.$

13. $\dfrac{\{f(1.1) - f(1)\}}{(1.1-1)} = \dfrac{\{(1.1)^2 - 1^2\}}{(1.1-1)} = (1.1 + 1) = 2.1.$

14. Highest prime factors of 12, 13, 14, 15, 16, 17 are 3, 13, 7, 5, 2, 17 respectively.

15. (i) For every $x \in R^+$, we have $\log_e x = R$. So, range $(f) = R$.

(ii) $f(x) = -2 \Rightarrow \log_e x = -2 \Rightarrow x = e^{-2}$. So, $\{x : x \in R^+ \text{ and } f(x) = -2\} = \{e^{-2}\}$.

(iii) $f(xy) = \log_e (xy) = \log_e x + \log_e y = f(x) + f(y)$.

16. (i) $f(x) = 2^x > 0$ for every $x \in R$.

If $x \in R^+$, there exists $\log_2 x$ such that $f(\log_2 x) = 2^{\log_2 x} = x$.

$\therefore$ range $(f) = R^+$.

(ii) $f(x) = 1 \Rightarrow 2^x = 1 = 2^0 \Rightarrow x = 0$. So, $\{x : f(x) = 1\} = \{0\}$.

(iii) $f(x + y) = 2^{x+y} = 2^x \times 2^y = f(x) \cdot f(y)$.

17. Dom $(f) = R$ and dom $(g) = C \Rightarrow$ dom $(f) \neq$ dom (g)

18. (i) Clearly, $f(x) = x^2 \geq 0$ for all $x \in R$.

(ii) $f(x) = x^2 + 1 \geq 1$ for all $x \in R$.

(iii) $\sin x \in [-1, 1]$.

19. (i) $x^2 + 1 = 10 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3.$

(ii) $x^2 + 1 = -3 \Rightarrow x^2 = -4$ and there is no real number whose square is –4.

REAL FUNCTIONS

REAL-VALUED FUNCTIONS *A function which has either R or one of its subsets as its range, is called a real-valued function.*

REAL FUNCTIONS *A function $f : X \to Y$ is called a real function, if $x \subseteq R$ and $y \subseteq R$.*

In general, real functions are described as general expressions or formulae, without mentioning their domains and co-domains. Following are some examples of real functions.

SOLVED EXAMPLES

EXAMPLE 1 *If $f(x) = 3x^3 - 5x^2 + 10$, find $f(x-1)$.*

SOLUTION We have, $f(x) = 3x^3 - 5x^2 + 10.$ … (i)

Replacing x by $(x-1)$ in (i), we get

$$\begin{aligned} f(x-1) &= 3(x-1)^3 - 5(x-1)^2 + 10 \\ &= 3\{x^3 - 1 - 3x(x-1)\} - 5(x^2 - 2x + 1) + 10 \\ &= 3(x^3 - 3x^2 + 3x - 1) - 5(x^2 - 2x + 1) + 10 \\ &= 3x^3 - 14x^2 + 19x + 2. \end{aligned}$$

$\therefore \quad f(x-1) = 3x^3 - 14x^2 + 19x + 2.$

EXAMPLE 2 *If $f(x) = x + \frac{1}{x}$, show that $\{f(x)\}^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$.*

SOLUTION We have, $f(x) = x + \frac{1}{x}$. … (i)

On cubing both sides of (i), we get

$$\begin{aligned} \{f(x)\}^3 &= x^3 + \frac{1}{x^3} + 3 \times x \times \frac{1}{x} \times \left(x + \frac{1}{x}\right) \\ &= \left(x^3 + \frac{1}{x^3}\right) + 3\left(\frac{1}{x} + x\right) \\ &= f(x^3) + 3f\left(\frac{1}{x}\right) \qquad \left[\because \ f\left(\frac{1}{x}\right) = \left\{\frac{1}{x} + \frac{1}{\frac{1}{x}}\right\} = \left(\frac{1}{x} + x\right)\right]. \end{aligned}$$

Hence, $\{f(x)\}^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$.

EXAMPLE 3 *If $f(x) = \frac{x-1}{x+1}$, $x \neq -1$ then show that $f\{f(x)\} = \frac{-1}{x}$, where $x \neq 0$.*

SOLUTION We have, $f(x) = \frac{x-1}{x+1}$, where $x \neq -1$.

$$\therefore \quad f\{f(x)\} = f\left(\frac{x-1}{x+1}\right) = \frac{\left\{\frac{x-1}{x+1} - 1\right\}}{\left\{\frac{x-1}{x+1} + 1\right\}}$$

$$= \frac{\{(x-1) - (x+1)\}}{(x+1)} \times \frac{(x+1)}{\{(x-1) + (x+1)\}} = \frac{-2}{2x} = \frac{-1}{x}.$$

Hence, $f\{f(x)\} = \frac{-1}{x}$, where $x \neq 0$.

EXAMPLE 4 *If $y = f(x) = \frac{ax - b}{bx - a}$ and $a^2 \neq b^2$ then prove that $x = f(y)$.*

SOLUTION We have, $y = f(x) = \dfrac{ax-b}{bx-a}$.

$$\therefore \quad f(y) = f\{f(x)\} = f\left(\frac{ax-b}{ax-a}\right) = \frac{\left\{a\left(\frac{ax-b}{bx-a}\right)-b\right\}}{\left\{b\left(\frac{ax-b}{bx-a}\right)-a\right\}}$$

$$= \frac{\{(a^2x-ab)-(b^2x-ab)\}}{(bx-a)} \times \frac{(bx-a)}{\{(abx-b^2)-(abx-a^2)\}}$$

$$= \frac{(a^2x-b^2x)}{(a^2-b^2)} = \frac{(a^2-b^2)x}{(a^2-b^2)} = x.$$

Hence, $x = f(y)$.

EXAMPLE 5 *If* $f(x) = \dfrac{x-1}{x+1}$ *then prove that* $f(2x) = \dfrac{3f(x)+1}{f(x)+3}$.

SOLUTION We have, $f(x) = \dfrac{x-1}{x+1}$.

$$\therefore \quad \frac{3f(x)+1}{f(x)+3} = \frac{3\left(\frac{x-1}{x+1}\right)+1}{\frac{(x-1)}{(x+1)}+3}$$

$$= \frac{(3x-3)+(x+1)}{(x+1)} \times \frac{(x+1)}{(x-1)+(3x+3)} = \frac{2x-1}{2x+1} = f(2x).$$

Hence, $f(2x) = \dfrac{3f(x)+1}{f(x)+3)}$.

EXAMPLE 6 *Let the functions f and g be defined by*

$$f(x) = (x-3) \text{ and } g(x) = \begin{cases} \dfrac{x^2-9}{x+3}, & \text{when } x \neq -3 \\ k, & \text{when } x = -3. \end{cases}$$

Find the value of k such that $f(x) = g(x)$ *for all* $x \in R$.

SOLUTION We have, $f(x) = g(x)$ for all $x \in R$.

$\therefore \quad f(-3) = g(-3)$

$\Rightarrow \quad (-3-3) = k \qquad [\because \ f(-3) = (-3-3) \text{ and } g(-3) = k]$

$\Rightarrow \quad k = -6.$

Hence, $k = -6$.

EXAMPLE 7 *If* $af(x) + bf\left(\dfrac{1}{x}\right) = \dfrac{1}{x}$, *where* $a \neq b$ *and* $x \neq 0$, *find* $f(x)$.

SOLUTION We have, $af(x) + bf\left(\dfrac{1}{x}\right) = \dfrac{1}{x}$. … (i)

Replacing x by $\dfrac{1}{x}$ in (i), we get

$$af\left(\frac{1}{x}\right) + bf(x) = x. \qquad \text{… (ii)}$$

Adding (i) and (ii), we get

$$a\left\{f(x)+f\left(\frac{1}{x}\right)\right\}+b\left\{f(x)+f\left(\frac{1}{x}\right)\right\}=\left(x+\frac{1}{x}\right)$$

$$\Rightarrow \quad (a+b)\left\{f(x)+f\left(\frac{1}{x}\right)\right\}=\left(x+\frac{1}{x}\right)$$

$$\Rightarrow \quad f(x)+f\left(\frac{1}{x}\right)=\frac{1}{(a+b)}\left(x+\frac{1}{x}\right). \qquad \ldots \text{(iii)}$$

On subtracting (ii) from (i), we get

$$a\left\{f(x)-f\left(\frac{1}{x}\right)\right\}-b\left\{f(x)-f\left(\frac{1}{x}\right)\right\}=\left(\frac{1}{x}-x\right)$$

$$\Rightarrow \quad (a-b)\left\{f(x)-f\left(\frac{1}{x}\right)\right\}=\left(\frac{1}{x}-x\right)$$

$$\Rightarrow \quad \left\{f(x)-f\left(\frac{1}{x}\right)\right\}=\frac{1}{(a-b)}\left(\frac{1}{x}-x\right). \qquad \ldots \text{(iv)}$$

On adding (iii) and (iv), we get

$$2f(x)=\left\{\frac{x}{(a+b)}-\frac{x}{(a-b)}\right\}+\left\{\frac{1}{(a+b)x}+\frac{1}{(a-b)x}\right\}$$

$$=\frac{(a-b)x-(a+b)x}{(a^2-b^2)}+\frac{(a-b)+(a+b)}{(a^2-b^2)x}$$

$$=\frac{-2bx^2+2d}{(a^2-b^2)x}=\frac{2}{(a^2-b^2)}\left(\frac{a}{x}-bx\right).$$

Hence, $f(x)=\frac{1}{(a^2-b^2)}\left(\frac{a}{x}-bx\right).$

EXERCISE 3B

1. If $f(x)=x^2-3x+4$ and $f(x)=f(2x+1)$, find the values of x.

2. If $f(x)=\frac{x-1}{x+1}$ then show that (i) $f\left(\frac{1}{x}\right)=-f(x)$ (ii) $f\left(\frac{-1}{x}\right)=\frac{-1}{f(x)}$.

3. If $f(x)=x^3-\frac{1}{x^3}$ then show that $f(x)+f\left(\frac{1}{x}\right)=0$.

4. If $f(x)=\frac{x+1}{x-1}$ then show that $f\{f(x)\}=x$.

5. If $f(x)=\frac{1}{(2x+1)}$ and $x\neq\frac{-1}{2}$ then prove that $f\{f(x)\}=\frac{2x+1}{2x+3}$, when it is given that $x\neq\frac{-3}{2}$.

6. If $f(x)=\frac{1}{(1-x)}$ then show that $f[f\{f(x)\}]=x$.

7. If $f(x)=\frac{2x}{(1+x^2)}$ then show that $f(\tan\theta)=\sin 2\theta$.

8. If $y=f(x)=\frac{3x+1}{5x-3}$, prove that $x=f(y)$.

ANSWERS (EXERCISE 3B)

1. $x = -1$ or $x = \frac{2}{3}$

HINTS TO SOME SELECTED QUESTIONS

1. $f(x) = f(2x+1) \Rightarrow x^2 - 3x + 4 = (2x+1)^2 - 3(2x+1) + 4$

$\Rightarrow x^2 - 3x + 4 = 4x^2 - 2x + 2 \Rightarrow 3x^2 + x - 2 = 0 \Rightarrow (x+1)(3x-2) = 0.$

5. $f(x) = \frac{1}{(2x+1)} \Rightarrow f\{f(x)\} = \frac{1}{\left\{2 \times \frac{1}{(2x+1)} + 1\right\}} = \frac{2x+1}{2x+3}.$

6. $f(x) = \frac{1}{(1-x)} \Rightarrow f\{f(x)\} = \frac{1}{\left\{1 - \frac{1}{(1-x)}\right\}} = \frac{1-x}{-x} = \frac{x-1}{x}$ [replace x by $\frac{1}{(1-x)}$]

$\Rightarrow f[f\{f(x)\}] = \frac{\left\{\frac{1}{(1-x)} - 1\right\}}{\frac{1}{(1-x)}} = \frac{x}{(1-x)} \times \frac{(1-x)}{1} = x.$

7. $f(\tan\theta) = \frac{2\tan\theta}{(1+\tan^2\theta)} = \frac{2\tan\theta}{\sec^2\theta} = \frac{2\sin\theta}{\cos\theta} \times \cos^2\theta = 2\sin\theta \cdot \cos\theta = \sin 2\theta.$

8. $f(y) = \frac{3y+1}{5y-3} = \frac{3\left(\frac{3x+1}{5x-3}\right)+1}{5\left(\frac{3x+1}{5x-3}\right)-3} = \frac{9x+3+5x-3}{15x+5-15x+9} = \frac{14x}{14} = x.$

PROBLEMS BASED ON DOMAINS AND RANGES OF REAL FUNCTIONS

SOLVED EXAMPLES

EXAMPLE 1 *Find the domain and the range of the real function,* $f(x) = \frac{1}{x+3}$.

SOLUTION We have, $f(x) = \frac{1}{(x+3)}$.

Clearly, $f(x)$ is defined for all real values of x except that at which $x + 3 = 0$, i.e., $x = -3$.

$\therefore$ dom $(f) = R - \{-3\}$.

Let $y = f(x)$. Then,

$$y = \frac{1}{x+3} \Rightarrow x + 3 = \frac{1}{y}$$

$$\Rightarrow x = \left(\frac{1}{y} - 3\right). \quad \ldots \text{(i)}$$

It is clear from (i) that x assumes real values for all real values of y except $y = 0$.

$\therefore$ range $(f) = R - \{0\}$.

Hence, dom $(f) = R - \{-3\}$ and range $(f) = R - \{0\}$.

EXAMPLE 2 *Find the domain and the range of the real function,* $f(x) = \dfrac{x-3}{x-5}$.

SOLUTION We have, $f(x) = \dfrac{x-3}{x-5}$.

Clearly, $f(x)$ is defined for all real values of x except that at which $x - 5 = 0$, i.e., $x = 5$.

$\therefore$ dom $(f) = R - \{5\}$.

Let $y = f(x)$. Then,

$$y = \frac{x-3}{x-5} \Rightarrow xy - 5y = x - 3$$

$$\Rightarrow x(y-1) = 5y - 3 \Rightarrow x = \frac{5y-3}{y-1}. \quad \ldots \text{(i)}$$

It is clear from (i) that x is not defined when $y - 1 = 0$, i.e., when $y = 1$.

$\therefore$ range $(f) = R - \{1\}$.

Hence, dom $(f) = R - \{5\}$ and range $(f) = R - \{1\}$.

EXAMPLE 3 *Find the domain and the range of the real function,* $f(x) = \dfrac{x^2+1}{x^2-1}$.

SOLUTION We have, $f(x) = \dfrac{x^2+1}{x^2-1}$.

Clearly, $f(x)$ is defined for all real values of x except those for which $x^2 - 1 = 0$, i.e., $x = \pm 1$.

$\therefore$ dom $(f) = R - \{-1, 1\}$.

Let $y = f(x)$. Then,

$$y = \frac{x^2+1}{x^2-1} \Rightarrow x^2y - y = x^2 + 1 \Rightarrow x^2(y-1) = (y+1)$$

$$\Rightarrow x^2 = \frac{y+1}{y-1} \Rightarrow x = \pm\sqrt{\frac{y+1}{y-1}}. \quad \ldots \text{(i)}$$

It is clear from (i) that x is not defined when $y - 1 = 0$ or when $\dfrac{y+1}{y-1} < 0$.

Now, $y - 1 = 0 \Rightarrow y = 1$. $\ldots$ (ii)

And $\dfrac{y+1}{y-1} < 0 \Rightarrow (y+1 > 0 \text{ and } y-1 < 0) \text{ or } (y+1 < 0 \text{ and } y-1 > 0)$

$$\Rightarrow (y > -1 \text{ and } y < 1) \text{ or } (y < -1 \text{ and } y > 1)$$

$$\Rightarrow -1 < y < 1. \quad \ldots \text{(iii)}$$

[$\because$ $y < -1$ and $y > 1$ is not possible]

Thus, x is not defined when $-1 < y \leq 1$. [using (ii) and (iii)]

$\therefore$ range $(f) = R - (-1, 1]$.

Hence, dom $(f) = R - \{-1, 1\}$ and range $(f) = R - (-1, 1]$.

EXAMPLE 4 *Find the domain of the real-valued function:*

$$f(x) = \frac{x^2 - x + 1}{x^2 - 5x + 4}.$$

SOLUTION We have, $f(x) = \dfrac{x^2 - x + 1}{x^2 - 5x + 4}$.

Clearly, $f(x)$ is defined for all real values of x except those at which $x^2 - 5x + 4 = 0$.

But, $x^2 - 5x + 4 = 0 \Rightarrow (x-1)(x-4) = 0 \Rightarrow x = 1 \text{ or } x = 4$.

$\therefore$ dom $(f) = R - \{1, 4\}$.

EXAMPLE 5 *Find the domain of each of the following real functions:*

(i) $f(x) = \sqrt{x-3}$ *(ii)* $g(x) = \sqrt{4 - x^2}$ *(iii)* $h(x) = \dfrac{1}{\sqrt{1-x}}$.

SOLUTION (i) We have, $f(x) = \sqrt{x-3}$.

Clearly, $f(x)$ is defined for all real values of x for which $x - 3 \geq 0$, i.e., $x \geq 3$.

$\therefore$ dom $(f) = [3, \infty)$.

(ii) We have, $g(x) = \sqrt{4 - x^2}$.

Clearly, $g(x)$ is defined for all real values of x for which $(4 - x^2) \geq 0$.

But, $(4 - x^2) \geq 0 \Rightarrow -(4 - x^2) \leq 0$

$\Rightarrow x^2 - 4 \leq 0 \Rightarrow (x+2)(x-2) \leq 0$

$\Rightarrow -2 \leq x \leq 2 \Rightarrow x \in [-2, 2]$.

$\therefore$ dom $(g) = [-2, 2]$.

(iii) We have, $h(x) = \dfrac{1}{\sqrt{1-x}}$.

Clearly, $h(x)$ is defined for all real values of x for which $(1 - x) > 0$.

But, $1 - x > 0 \Rightarrow 1 > x \Rightarrow x < 1 \Rightarrow x \in (-\infty, 1)$.

$\therefore$ dom $(h) = (-\infty, 1)$.

EXAMPLE 6 *Find the domain of the function,* $f(x) = \sqrt{3-x} + \dfrac{1}{\sqrt{x^2 - 1}}$.

SOLUTION We have, $f(x) = \sqrt{3-x} + \dfrac{1}{\sqrt{x^2-1}}$.

Clearly, $f(x)$ is defined for all real values of x for which

$3 - x \geq 0$ and $x^2 - 1 > 0$

$\Rightarrow$ $x - 3 \leq 0$ and $(x+1)(x-1) > 0$

$\Rightarrow$ $x \leq 3$ and $(x < -1 \text{ or } x > 1)$

$\Rightarrow$ $(x \leq 3 \text{ and } x < -1) \text{ or } (x \leq 3 \text{ and } x > 1)$

$\Rightarrow \quad (x < -1) \text{ or } (1 < x \leq 3)$

$\Rightarrow \quad x \in (-\infty, -1) \cup (1, 3].$

$\therefore \quad \text{dom}\,(f) = (-\infty, -1) \cup (1, 3].$

EXAMPLE 7 *Find the range of each of the following functions:*

(i) $f(x) = 2 - 3x,\ x \in R \text{ and } x > 0$ *(ii)* $g(x) = x^2 + 2,\ x \in R.$

SOLUTION (i) We have, $f(x) = 2 - 3x$, where $x \in R$ and $x > 0$.

Now, $x > 0 \Rightarrow 3x > 0 \Rightarrow -3x < 0$

$\Rightarrow -3x + 2 < 0 + 2 \Rightarrow 2 - 3x < 2$

$\Rightarrow f(x) < 2 \Rightarrow f(x) \in (-\infty, 2).$

Hence, range $(f) = (-\infty, 2)$.

(ii) We have, $g(x) = x^2 + 2,\ x \in R.$

Now, $x \in R \Rightarrow x^2 \geq 0 \Rightarrow x^2 + 2 \geq 0 + 2$

$\Rightarrow x^2 + 2 \geq 2 \Rightarrow g(x) \geq 2$

$\Rightarrow g(x) \in [2, \infty).$

Hence, range $(g) = [2, \infty)$.

EXAMPLE 8 *Find the domain and the range of the function,* $f(x) = \dfrac{x-2}{x-3}$.

SOLUTION We have, $f(x) = \dfrac{x-2}{x-3}$.

Clearly, $f(x)$ is defined for all real values of x for which $x - 3 \neq 0$, i.e., $x \neq 3$.

$\therefore \quad \text{dom}\,(f) = R - \{3\}.$

Let $y = f(x)$. Then,

$$y = \frac{x-2}{x-3} \Rightarrow xy - 3y = x - 2$$

$$\Rightarrow x(y-1) = 3y - 2$$

$$\Rightarrow x = \frac{3y-2}{y-1}. \qquad \ldots (i)$$

It follows from (i) that x assumes real values for all y except that for which $y - 1 = 0$, i.e., $y = 1$.

$\therefore \quad \text{range}\,(f) = R - \{1\}.$

Hence, dom $(f) = R - \{3\}$ and range $(f) = R - \{1\}$.

EXAMPLE 9 *Find the domain and the range of the real function,* $f(x) = \sqrt{x-3}$.

SOLUTION We have, $f(x) = \sqrt{x-3}$.

Clearly, $f(x)$ is defined for all real values of x for which $x - 3 \geq 0$, i.e., $x \geq 3$.

$\therefore \quad \text{dom}\,(f) = [3, \infty).$

Also, $x \geq 3 \Rightarrow f(x) = \sqrt{x-3} \geq 0.$

$\therefore \quad \text{range}\,(f) = [0, \infty).$

Hence, dom $(f) = [3, \infty)$ and range $(f) = [0, \infty)$.

EXAMPLE 10 *Find the domain and the range of each of the functions given below.*

(i) $f(x) = |x-1|$ *(ii)* $g(x) = -|x|$

SOLUTION (i) We have, $f(x) = |x-1|$.

Clearly, $f(x)$ is defined for all $x \in R$. So, dom $(f) = R$.

For all $x \in R$, we have

$$|x-1| \geq 0 \Rightarrow f(x) \geq 0.$$

$\therefore$ range $(f) = [0, \infty)$.

Hence, dom $(f) = R$ and range $(f) = [0, \infty)$.

(ii) We have, $g(x) = -|x|$.

Clearly, $g(x)$ is defined for all $x \in R$. So, dom $(g) = R$.

For all $x \in R$, we have

$$|x| \geq 0 \Rightarrow -|x| \leq 0 \Rightarrow g(x) \leq 0.$$

$\therefore$ range $(g) = (-\infty, 0]$.

Hence, dom $(g) = R$ and range $(g) = (-\infty, 0]$.

EXAMPLE 11 *Find the domain and the range of the real function,* $f(x) = \dfrac{1}{\sqrt{x-2}}$.

SOLUTION We have, $f(x) = \dfrac{1}{\sqrt{x-2}}$.

Clearly, $f(x)$ is defined for all real values of x for which $x-2 > 0$, i.e., $x > 2$.

$\therefore$ dom $(f) = (2, \infty)$.

For any real value of $x > 2$, we have

$$x > 2 \Rightarrow x-2 > 0 \Rightarrow \sqrt{x-2} > 0$$

$$\Rightarrow \frac{1}{\sqrt{x-2}} > 0 \Rightarrow f(x) > 0.$$

$\therefore$ range $(f) = (0, \infty)$.

Hence, dom $(f) = (2, \infty)$ and range $(f) = (0, \infty)$.

EXAMPLE 12 *Find the domain and the range of the real function,* $f(x) = \sqrt{9-x^2}$.

SOLUTION We have, $f(x) = \sqrt{9-x^2}$.

Clearly, $f(x)$ is defined for all real values of x for which $(9-x^2) \geq 0$.

And, $9-x^2 \geq 0 \Rightarrow x^2-9 \leq 0 \Rightarrow (x-3)(x+3) \leq 0$

$$\Rightarrow -3 \leq x \leq 3 \Rightarrow x \in [-3, 3].$$

$\therefore$ dom $(f) = [-3, 3]$.

Let $y = f(x)$. Then,

$$y = \sqrt{9-x^2} \Rightarrow y^2 = 9-x^2 \Rightarrow x^2 = 9-y^2 \Rightarrow x = \sqrt{9-y^2}.$$

Clearly, x will take real values only when $9-y^2 \geq 0$.

Now, $9-y^2 \geq 0 \Rightarrow y^2-9 \leq 0 \Rightarrow (y+3)(y-3) \leq 0$

$$\Rightarrow -3 \leq y \leq 3 \Rightarrow y \in [-3, 3]$$

$\Rightarrow\ y \in [0, 3]$ $\{\because\ y = \sqrt{9 - x^2} \geq 0$ for all $x \in [-3, 3]\}$.

$\therefore$ range $(f) = [0, 3]$.

Hence, dom $(f) = [-3, 3]$ and range $(f) = [0, 3]$.

EXAMPLE 13 *Find the domain and the range of the real function,* $f(x) = \dfrac{x}{(1 + x^2)}$.

SOLUTION We have, $f(x) = \dfrac{x}{(1 + x^2)}$.

Clearly, $f(x)$ is defined for all $x \in R$. So, dom $(f) = R$.

Let $y = f(x)$. Then,

$$y = \frac{x}{(1 + x^2)} \Rightarrow x^2 y - x + y = 0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{1 - 4y^2}}{2y}. \qquad \ldots \text{(i)}$$

It is clear from (i) that x will take real values, when

$(1 - 4y^2) \geq 0$ and $y \neq 0$

$\Rightarrow$ $(4y^2 - 1) \leq 0$ and $y \neq 0$

$\Rightarrow$ $(2y + 1)(2y - 1) \leq 0$ and $y \neq 0$

$\Rightarrow$ $\left(y + \frac{1}{2}\right)\left(y - \frac{1}{2}\right) \leq 0$ and $y \neq 0$

$\Rightarrow$ $-\frac{1}{2} \leq y \leq \frac{1}{2}$ and $y \neq 0$

$\Rightarrow$ $y \in \left[\frac{-1}{2}, \frac{1}{2}\right] - \{0\}$.

Also, $x = 0 \Rightarrow y = 0$.

$\therefore$ range $(f) = \left[\frac{-1}{2}, \frac{1}{2}\right]$.

Hence, dom $(f) = R$ and range $(f) = \left[\frac{-1}{2}, \frac{1}{2}\right]$.

EXAMPLE 14 *Find the domain and the range of the real function,* $f(x) = \dfrac{3}{(2 - x^2)}$.

SOLUTION We have, $f(x) = \dfrac{3}{(2 - x^2)}$.

Clearly, $f(x)$ is defined for all real values of x except those for which $2 - x^2 = 0$, i.e., $x = \pm\sqrt{2}$.

$\therefore$ dom $(f) = R - \{-\sqrt{2}, \sqrt{2}\}$.

Let $y = f(x)$. Then,

$$y = \frac{3}{(2 - x^2)} \Rightarrow 2y - x^2 y = 3 \Rightarrow x^2 y = 2y - 3$$

$$\Rightarrow x^2 = \frac{2y - 3}{y} \Rightarrow x = \pm\sqrt{\frac{2y - 3}{y}}. \qquad \ldots \text{(i)}$$

It is clear from (i) that x will take real values only when $\frac{2y-3}{y} \geq 0$.

Now, $\frac{2y-3}{y} \geq 0 \Leftrightarrow (2y-3 \leq 0 \text{ and } y < 0) \text{ or } (2y-3 \geq 0 \text{ and } y > 0)$

$\Leftrightarrow \left(y \leq \frac{3}{2} \text{ and } y < 0\right) \text{ or } \left(y \geq \frac{3}{2} \text{ and } y > 0\right)$

$\Leftrightarrow (y < 0) \text{ or } \left(y \geq \frac{3}{2}\right)$

$\Leftrightarrow y \in (-\infty, 0) \text{ or } y \in \left[\frac{3}{2}, \infty\right)$

$\Leftrightarrow y \in (-\infty, 0) \cup \left[\frac{3}{2}, \infty\right)$.

$\therefore$ range $(f) = (-\infty, 0) \cup \left[\frac{3}{2}, \infty\right)$.

Hence, dom $(f) = R - \{-\sqrt{2}, \sqrt{2}\}$ and range $(f) = (-\infty, 0) \cup \left[\frac{3}{2}, \infty\right)$.

EXAMPLE 15 *Find the domain and the range of the real function* $f = \left\{\left(x, \frac{x^2}{x^2+1}\right) : x \in R\right\}$ *from R into R.*

SOLUTION We have, $f(x) = \frac{x^2}{x^2+1}, x \in R$.

Clearly, $x^2 + 1 \neq 0$ for any real value of x.

$\therefore$ $f(x)$ is defined for all real values of x.

$\therefore$ dom $(f) = R$.

Let $y = f(x)$. Then,

$$y = \frac{x^2}{(x^2+1)} \Rightarrow x^2y + y = x^2 \Rightarrow x^2(1-y) = y$$

$$\Rightarrow x^2 = \frac{y}{1-y} \Rightarrow x = \pm\sqrt{\frac{y}{(1-y)}}. \qquad \ldots \text{(i)}$$

It is clear from (i) that x will take real values only when $\frac{y}{(1-y)} \geq 0$.

Now, $\frac{y}{(1-y)} \geq 0 \Leftrightarrow (y \leq 0 \text{ and } 1-y < 0) \text{ or } (y \geq 0 \text{ and } 1-y > 0)$

$\Leftrightarrow (y \leq 0 \text{ and } y > 1) \text{ or } (y \geq 0 \text{ and } y < 1)$

$\Leftrightarrow (y \geq 0 \text{ and } y < 1)$

[$\because$ $y \leq 0$ and $y > 1$ is not possible]

$\Leftrightarrow y \in [0, 1)$.

$\therefore$ range $(f) = [0, 1)$.

Hence, dom $(f) = R$ and range $(f) = [0, 1)$.

EXAMPLE 16 *Find the domain and the range of the function*

$$f = \left\{\left(x, \frac{1}{1-x^2}\right) : x \in R \text{ and } x \neq \pm 1\right\}.$$

SOLUTION We have, $f(x) = \dfrac{1}{(1-x^2)}, x \in R.$

Clearly, $f(x)$ is defined for all real values of x for which $(1-x^2) \neq 0.$

Now, $(1-x^2) = 0 \Rightarrow (1+x)(1-x) = 0 \Rightarrow x = -1$ or $x = +1.$

Thus, $f(x)$ is defined for all values of $x \in R$ except $\pm 1.$

$\therefore$ dom $(f) = R - \{-1, 1\}.$

Let $y = f(x)$. Then,

$$y = \frac{1}{1-x^2} \Rightarrow y - x^2 y = 1 \Rightarrow x^2 y = y - 1$$

$$\Rightarrow x^2 = \frac{y-1}{y} \Rightarrow x = \pm\sqrt{\frac{y-1}{y}}. \qquad \ldots \text{(i)}$$

It is clear from (i) that x will take real values only when $\dfrac{y-1}{y} \geq 0.$

Now, $\dfrac{y-1}{y} \geq 0 \Leftrightarrow (y-1 \leq 0 \text{ and } y < 0)$ or $(y-1 \geq 0 \text{ and } y > 0)$

$$\Leftrightarrow (y \leq 1 \text{ and } y < 0) \text{ or } (y \geq 1 \text{ and } y > 0)$$

$$\Leftrightarrow (y < 0) \text{ or } (y \geq 1)$$

$$\Leftrightarrow y \in (-\infty, 0) \text{ or } [1, \infty).$$

$\therefore$ range $(f) = (-\infty, 0) \cup [1, \infty).$

Hence, dom $(f) = R - \{-1, 1\}$ and range $(f) = (-\infty, 0) \cup [1, \infty).$

EXAMPLE 17 *Find the domain and the range of the real function,* $f(x) = \dfrac{x^2-25}{x-5}.$

SOLUTION We have, $f(x) = \dfrac{x^2-25}{x-5}.$

Clearly, $f(x)$ is defined for all real values of x for which $x - 5 \neq 0,$ i.e., $x \neq 5.$

$\therefore$ dom $(f) = R - \{5\}.$

Let $y = f(x)$. Then,

$$y = \frac{x^2-25}{x-5} \Rightarrow y = x + 5, \text{ when } x - 5 \neq 0$$

$$\Rightarrow y = x + 5, \text{ when } x \neq 5$$

$$\Rightarrow y \neq 5 + 5 \Rightarrow y \neq 10.$$

Then, y can be assigned any real value except 10.

$\therefore$ range $(f) = R - \{10\}.$

Hence, dom $(f) = R - \{5\}$ and range $(f) = R - \{10\}.$

EXAMPLE 18 *Find the domain and the range of the real function,* $f(x) = \dfrac{3-x}{x-3}.$

SOLUTION We have, $f(x) = \dfrac{3-x}{x-3}.$

Clearly, $f(x)$ is defined for all real values of x for which $x-3\neq 0$, i.e., $x\neq 3$.

$\therefore$ dom $(f)=R-\{3\}$.

Let $y=f(x)$. Then,

$$y=\frac{3-x}{x-3} \Rightarrow y=-1, \text{ when } x-3\neq 0$$

$$\Rightarrow y=-1, \text{ when } x\neq 3.$$

$\therefore$ range $(f)=\{-1\}$.

Hence, dom $(f)=R-\{3\}$ and range $(f)=\{-1\}$.

EXAMPLE 19 *Find the domain and the range of the real function,* $f(x)=\dfrac{|x-3|}{(x-3)}$.

SOLUTION We have, $f(x)=\dfrac{|x-3|}{(x-3)}$.

Clearly, $f(x)$ is defined for all real values of x for which $x-3\neq 0$, i.e., $x\neq 3$.

$\therefore$ dom $(f)=R-\{3\}$.

Now, when $x\neq 3$, we have

$$f(x)=\begin{cases}1, \text{ when } x-3>0 & \{\because |x-3|=(x-3)\}\\ -1, \text{ when } x-3<0 & \{\because |x-3|=-(x-3)\}\end{cases}$$

$\therefore$ range $(f)=\{1,-1\}$.

Hence, dom $(f)=R-\{3\}$ and range $(f)=\{1,-1\}$.

EXAMPLE 20 *Find the domain of the real function,* $f(x)=\dfrac{1}{\sqrt{x+|x|}}$.

SOLUTION We have, $f(x)=\dfrac{1}{\sqrt{x+|x|}}$.

Now, $\quad |x|=\begin{cases}x, \text{ when } x\geq 0\\ -x, \text{ when } x<0\end{cases}$

$\Rightarrow \quad x+|x|=\begin{cases}x+x, \text{ when } x\geq 0\\ x-x, \text{ when } x<0\end{cases}$

$\Rightarrow \quad x+|x|=\begin{cases}2x, \text{ when } x\geq 0\\ 0, \text{ when } x<0\end{cases}$

$\Rightarrow \quad x+|x|>0$, when $x>0$

$\Rightarrow \quad f(x)=\dfrac{1}{\sqrt{x+|x|}}$ assumes real values only when $x+|x|>0$ and this happens only when $x>0$.

$\therefore$ dom $(f)=(0,\infty)$.

EXAMPLE 21 *Show that* $f(x)=\dfrac{1}{\sqrt{x-|x|}}$ *is not defined for any* $x\in R$. *How will you define dom* (f) *and range* (f)?

SOLUTION We have, $f(x)=\frac{1}{\sqrt{x-|x|}}$.

Now, $|x|=\begin{cases}x, \text{ when } x\geq 0\\ -x, \text{ when } x<0\end{cases}$

$\Rightarrow \quad x-|x|=\begin{cases}x-x, \text{ when } x\geq 0\\ x+x, \text{ when } x<0\end{cases}$

$\Rightarrow \quad x-|x|=\begin{cases}0, \text{ when } x\geq 0\\ 2x, \text{ when } x<0\end{cases}$

$\Rightarrow \quad x-|x|\leq 0$ for all $x\in R$

$\Rightarrow \quad \frac{1}{\sqrt{x-|x|}}$ is not defined for any $x\in R$.

$\therefore \quad$ dom $(f)=\phi$ and range $(f)=\phi$.

GREATEST INTEGER FUNCTION (STEP FUNCTION)

The function $f:R\to R: f(x)=[x]$, where $[x]$ denotes the greatest integer less than or equal to x, is called the Greatest Integer Function.

EXAMPLES $[3.65]=3, [3.02]=3, [5]=5, [-2.6]=-3, [-4]=-4$.

NOTE $x-[x]=0$ for $x\in Z$ and $0<x-[x]<1$ for $x\in R-Z$.

EXAMPLE 22 *Find the domain and the range of the real function,* $f(x)=\frac{1}{\sqrt{x+[x]}}$.

SOLUTION We have, $f(x)=\frac{1}{\sqrt{x+[x]}}$.

We know that

$\begin{cases}x+[x]>0 \text{ for all } x>0\\ x+[x]=0, \text{ when } x=0\\ x+[x]<0 \text{ for all } x<0.\end{cases}$

$\therefore \quad f(x)=\frac{1}{\sqrt{x+[x]}}$ is defined only when $x+[x]>0$ and this happens only when $x>0$.

$\therefore \quad$ dom $(f)=(0,\infty)$.

Let $y=f(x)$. Then,

$$y=\frac{1}{\sqrt{x+[x]}} \Rightarrow \sqrt{x+[x]}=\frac{1}{y}. \qquad \ldots (i)$$

Now, $x>0 \Rightarrow x+[x]>0 \Rightarrow \sqrt{x+[x]}>0 \Rightarrow \frac{1}{y}>0 \Rightarrow y>0$.

$\therefore \quad$ range $(f)=(0,\infty)$.

Hence, dom $(f)=(0,\infty)$ and range $(f)=(0,\infty)$.

EXAMPLE 23 *Find the domain and the range of the real function,* $f(x)=\frac{1}{\sqrt{x-[x]}}$.

SOLUTION We have, $f(x)=\frac{1}{\sqrt{x-[x]}}$.

We know that

$0 \le x-[x]<1$ for all $x \in R$

and $x-[x]=0$ for all $x \in Z$.

$\therefore \quad 0<x-[x]<1$ for all $x \in R-Z$

$\Rightarrow \quad f(x)=\frac{1}{\sqrt{x-[x]}}$ exists for all $x \in R-Z$

$\Rightarrow \quad \text{dom}\,(f)=R-Z$.

Also, $0<x-[x]<1$ for all $x \in R-Z$

$\Rightarrow \quad 0<\sqrt{x-[x]}<1$ for all $x \in R-Z$

$\Rightarrow \quad 1<\frac{1}{\sqrt{x-[x]}}<\infty$ for all $x \in R-Z$

$\Rightarrow \quad 1<f(x)<\infty$ for all $x \in R-Z$

$\Rightarrow \quad \text{range}\,(f)=(1,\infty)$.

Hence, dom $(f)=R-Z$ and range $(f)=(1,\infty)$.

EXERCISE 3C

1. Find the domain of each of the following real functions:

(i) $f(x)=\frac{3x+5}{x^2-9}$ (ii) $f(x)=\frac{2x-3}{x^2+x-2}$

(iii) $f(x)=\frac{x^2-2x+1}{x^2-8x+12}$ (iv) $f(x)=\frac{x^3-8}{x^2-1}$

Find the domain and the range of each of the following real functions:

2. $f(x)=\frac{1}{x}$ **3.** $f(x)=\frac{1}{(x-5)}$ **4.** $f(x)=\frac{x-3}{2-x}$

5. $f(x)=\frac{3x-2}{x+2}$ **6.** $f(x)=\frac{x^2-16}{x-4}$ **7.** $f(x)=\frac{1}{\sqrt{2x-3}}$

8. $f(x)=\frac{ax-b}{cx-d}$ **9.** $f(x)=\sqrt{3x-5}$ **10.** $f(x)=\sqrt{\frac{x-5}{3-x}}$

11. $f(x)=\frac{1}{\sqrt{x^2-1}}$ **12.** $f(x)=1-|x-2|$ **13.** $f(x)=\frac{|x-4|}{x-4}$

14. $f(x)=\frac{x^2-9}{x-3}$ **15.** $f(x)=\frac{1}{2-\sin 3x}$

ANSWERS (EXERCISE 3C)

1. (i) $R-\{3,-3\}$ (ii) $R-\{1,-2\}$ (iii) $R-\{2,6\}$ (iv) $R-\{1,-1\}$

2. dom $(f)=R-\{0\}$, range $(f)=R-\{0\}$

3. dom $(f)=R-\{5\}$, range $(f)=R-\{0\}$

4. dom $(f) = R - \{2\}$, range $(f) = R - \{-1\}$

5. dom $(f) = R - \{-2\}$, range $(f) = R - \{3\}$

6. dom $(f) = R - \{4\}$, range $(f) = R - \{8\}$

7. dom $(f) = R - \left(-\infty, \frac{3}{2}\right]$, range $(f) = R - \{0\}$

8. dom $(f) = R - \left\{\frac{d}{c}\right\}$, range $(f) = R - \left\{\frac{a}{c}\right\}$

9. dom $(f) = \left[\frac{5}{3}, \infty\right)$, range $(f) = R - [0, \infty)$

10. dom $(f) = (3, 5]$, range $(f) = [0, \infty)$

11. dom $(f) = (-\infty, -1) \cup (1, \infty)$, range $(f) = R - \{0\}$

12. dom $(f) = R$, range $(f) = (-\infty, 1]$

13. dom $(f) = R - \{4\}$, range $(f) = \{-1, 1\}$

14. dom $(f) = R - \{3\}$, range $(f) = R - \{6\}$

15. dom $(f) = R$, range $(f) = \left[\frac{1}{3}, 1\right]$

HINTS TO SOME SELECTED QUESTIONS

2. $f(x) = \frac{1}{x}$ is not defined when $x = 0$. So, dom $(f) = R - \{0\}$.

Let $y = f(x)$. Then, $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$.

$\therefore$ x is not defined when $y = 0$. So, range $(f) = R - \{0\}$.

3. $f(x) = \frac{1}{(x-5)}$ is not defined when $x = 5$. So, dom $(f) = R - \{5\}$.

Let $y = f(x)$. Then, $y = \frac{1}{(x-5)} \Rightarrow x - 5 = \frac{1}{y} \Rightarrow x = \left(\frac{1}{y} + 5\right)$.

$\therefore$ x is not defined when $y - 0$. So, range $(f) - R - \{0\}$.

4. $f(x) = \frac{x-3}{2-x}$ is not defined when $x = 2$. So, dom $(f) = R - \{2\}$.

Let $y = f(x)$. Then,

$$y = \frac{x-3}{2-x} \Rightarrow 2y - xy = x - 3 \Rightarrow xy + x = 2y + 3$$

$$\Rightarrow x(y+1) = 2y + 3 \Rightarrow x = \frac{2y+3}{y+1}.$$

$\therefore$ x is not defined when $y = -1$. So, range $(f) = R - \{-1\}$.

5. $f(x) = \frac{3x-2}{x+2}$ is not defined when $x = -2$. So, dom $(f) = R - \{-2\}$.

Let $y = f(x)$. Then,

$$y = \frac{3x-2}{x+2} \Rightarrow xy + 2y = 3x - 2 \Rightarrow 3x - xy = 2y + 2$$

$$\Rightarrow x(3-y) = 2y + 2 \Rightarrow x = \frac{2y+2}{3-y}. \quad \ldots \text{(i)}$$

It follows from (i) that x is not defined when $y = 3$.

$\therefore$ range $(f) = R - \{3\}$.

6. $f(x)=\dfrac{x^2-16}{x-4}$ is not defined when $x=4$. So, dom $(f)=R-\{4\}$.

Let $y=f(x)$. Then,

$$y=\frac{x^2-16}{x-4}=(x+4), \text{ when } x\neq 4$$

$\Rightarrow \quad y\neq(4+4) \Rightarrow y\neq 8$. So, range $(f)=R-\{8\}$.

7. $f(x)=\dfrac{1}{\sqrt{2x-3}}$ is not defined when $2x-3\leq 0$, i.e., when $x\leq\dfrac{3}{2}$.

$\therefore \quad$ dom $(f)=R-\left(-\infty,\dfrac{3}{2}\right]$.

Let $y=f(x)$. Then,

$$y=\frac{1}{\sqrt{2x-3}} \Rightarrow y^2=\frac{1}{(2x-3)} \Rightarrow (2x-3)=\frac{1}{y^2} \Rightarrow x=\frac{1}{2}\left(\frac{1}{y^2}+3\right).$$

$\therefore \quad x$ is not defined when $y=0$. So, range $(f)=R-\{0\}$.

8. $f(x)=\dfrac{ax-b}{cx-d}$ is not defined when $cx-d=0 \Rightarrow x=\dfrac{d}{c}$.

$\therefore \quad$ dom $(f)=R-\left\{\dfrac{d}{c}\right\}$.

Let $y=f(x)$. Then,

$$y=\frac{ax-b}{cx-d} \Rightarrow cxy-dy=ax-b \Rightarrow cxy-ax=dy-b$$

$$\Rightarrow x(cy-a)=dy-b \Rightarrow x=\frac{dy-b}{cy-a}.$$

Clearly, x is not defined when $cy-a=0 \Rightarrow y=\dfrac{a}{c}$.

$\therefore \quad$ range $(f)=R-\left\{\dfrac{a}{c}\right\}$.

9. $f(x)=\sqrt{3x-5}$ is defined only when $3x-5\geq 0$, i.e., when $3x-5\geq 0$, i.e., when $x\geq\dfrac{5}{3}$.

$\therefore \quad$ dom $(f)=\left[\dfrac{5}{3},\infty\right)$.

$x\geq\dfrac{5}{3} \Rightarrow f(x)\geq 0 \Rightarrow$ range $(f)=R-[0,\infty)$.

10. $f(x)=\sqrt{\dfrac{x-5}{3-x}}$ is defined only when $\dfrac{x-5}{3-x}\geq 0$.

Now, $\dfrac{x-5}{3-x}=0 \Rightarrow x-5=0 \Rightarrow x=5$.

And, $\dfrac{x-5}{3-x}>0 \Rightarrow (x-5<0 \text{ and } 3-x<0) \text{ or } (x-5>0 \text{ and } 3-x>0)$

$$\Rightarrow (x<5 \text{ and } 3<x) \text{ or } (x>5 \text{ and } 3>x)$$
$$\Rightarrow (3<x \text{ and } x<5) \text{ or } (x>5 \text{ and } x<3)$$
$$\Rightarrow 3<x<5 \quad [\because\ x>5 \text{ and } x<3 \text{ cannot hold}].$$

Thus, $f(x)$ is defined when $3<x\leq 5$.

$\therefore \quad$ dom $(f)=(3,5]$.

Clearly, $f(x)=\sqrt{\dfrac{x-5}{3-x}}\geq 0$.

Let $y=f(x)$. Then,

$$y=\sqrt{\frac{x-5}{3-x}} \Rightarrow y^2=\frac{x-5}{3-x} \Rightarrow 3y^2-xy^2=x-5$$

$$\Rightarrow x+xy^2=3y^2+5 \Rightarrow x(1+y^2)=(3y^2+5) \Rightarrow x=\frac{3y^2+5}{1+y^2}. \quad \ldots \text{(i)}$$

It follows from (i) that x is defined for each real value of y.

$\therefore$ range $(f)=[0,\infty)$ $\quad [\because f(x)\geq 0]$.

11. $f(x)=\dfrac{1}{\sqrt{x^2-1}}$ is defined only when $x^2-1>0$.

Now, $x^2-1>0 \Rightarrow (x+1)(x-1)>0 \Rightarrow (x<-1)$ or $(x>1)$

$\Rightarrow x\in(-\infty,-1)$ or $x\in(1,\infty)$.

$\therefore$ dom $(f)=(-\infty,-1)\cup(1,\infty)$.

Let $y=f(x)$. Then,

$$y=\frac{1}{\sqrt{x^2-1}} \Rightarrow y^2=\frac{1}{(x^2-1)} \Rightarrow x^2-1=\frac{1}{y^2} \Rightarrow x=\sqrt{\frac{1}{y^2}+1}=\sqrt{\frac{1+y^2}{y^2}}.$$

Clearly, x is not defined when $y=0$.

$\therefore$ range $(f)=R-\{0\}$.

12. $f(x)=1-|x-2|$ is defined for all $x\in R$. So, dom $(f)=R$.

Now, $0\leq|x-2|<\infty \Rightarrow |x-2|\geq 0$ and $|x-2|<\infty$

$\Rightarrow -|x-2|\leq 0$ and $-|x-2|>-\infty$

$\Rightarrow 1-|x-2|\leq 1$ and $1-|x-2|>-\infty$

$\Rightarrow -\infty<1-|x-2|\leq 1 \Rightarrow f(x)\in(-\infty,1]$.

$\therefore$ range $(f)=(-\infty,1]$.

13. $f(x)=\dfrac{|x-4|}{x-4}$ is not defined at $x=4$. So, dom $(f)=R-\{4\}$.

Now, when $x\neq 4$, we have

$$f(x)=\begin{cases}1, \text{ when } x-4>0 & \{\because |x-4|=x-4\}\\ -1, \text{ when } x-4<0 & \{\because |x-4|=-(x-4)\}.\end{cases}$$

$\therefore$ range $(f)=[-1,1]$.

14. $f(x)=\dfrac{x^2-9}{x-3}$ is defined for all real values of x for which $x\neq 3$.

$\therefore$ dom $(f)=R-\{3\}$.

Let $y=f(x)$. Then,

$$y=\frac{x^2-9}{x-3} \Rightarrow y=x+3, \text{ when } x\neq 3$$

$$\Rightarrow y\neq 3+3 \Rightarrow y\neq 6.$$

$\therefore$ range $(f)=R-\{6\}$.

15. Clearly, $-1\leq -\sin 3x\leq 1$ for all $x\in R$

$\Rightarrow$ $2+(-1)\leq 2+(-\sin 3x)\leq 2+1$ for all $x\in R$

$\Rightarrow$ $1\leq 2-\sin 3x\leq 3$ for all $x\in R$

$\Rightarrow$ $2-\sin 3x\neq 0$ for any $x\in R$.

$\therefore$ $f(x)=\dfrac{1}{(2-\sin 3x)}$ is defined for all $x\in R$.

We have, $2 - \sin 3x \geq 1$ and $2 - \sin 3x \leq 3$

$\therefore \quad \frac{1}{2 - \sin 3x} \leq 1$ and $\frac{1}{2 - \sin 3x} \geq \frac{1}{3} \Rightarrow \frac{1}{3} \leq \frac{1}{2 - \sin 3x} \leq 1.$

$\therefore \quad \text{range}(f) = \left[\frac{1}{3}, 1\right].$

GRAPHS OF SOME STANDARD REAL FUNCTIONS

1. IDENTITY FUNCTION

The function $f : R \to R : f(x) = x$ for all $x \in R$ is called an *identity function* on *R*.

Dom $(f) = R$ and range $(f) = R$.

EXAMPLE 1 *Draw the graph of the identity function* $f : R \to R : f(x) = x$ *for all* $x \in R$.

SOLUTION Here $f(x) = x$ for all $x \in R$.

We may prepare its table as under.

x	–2	–1	0	1	2	3
$f(x) = x$	–2	–1	0	1	2	3

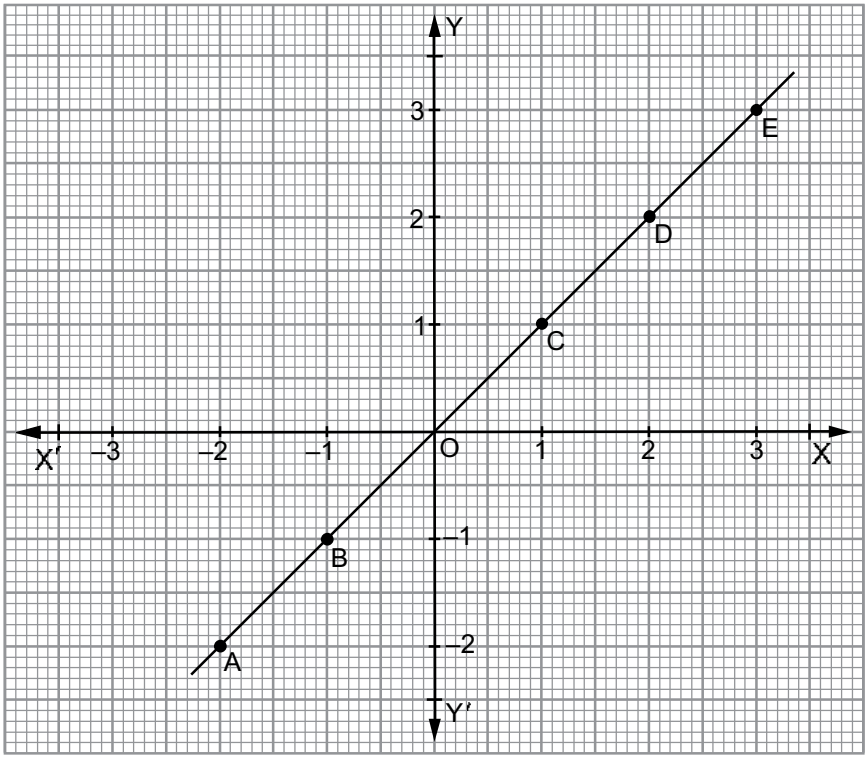
On a graph paper, we draw $X'OX$ and YOY' as the *x*-axis and the *y*-axis respectively.

We take 10 small divisions = 1 unit.

Now, on this graph paper, we plot the points

$A(-2, -2), B(-1, -1), O(0, 0), C(1, 1), D(2, 2)$ and $E(3, 3)$.

Join these points successively to obtain the straight line *ABOCDE* as the required graph line, given below.



Graph of the identity function, $f(x) = x$

2. CONSTANT FUNCTION

Let c be a fixed real number.

Then, the function $f : R \to R : f(x) = c$ for all $x \in R$ is called the *constant function*.

For the constant function $f(x) = c$ for all $x \in R$, we have

dom $(f) = R$ and range $(f) = \{c\}$.

EXAMPLE 2 *Draw the graph of each of the following constant functions:*

(i) $f(x) = 2$ *for all* $x \in R$

(ii) $f(x) = 0$ *for all* $x \in R$

(iii) $f(x) = -2$ *for all* $x \in R$

SOLUTION (i) Let $f(x) = 2$ for all $x \in R$.

Then, dom $(f) = R$ and range $(f) = \{2\}$.

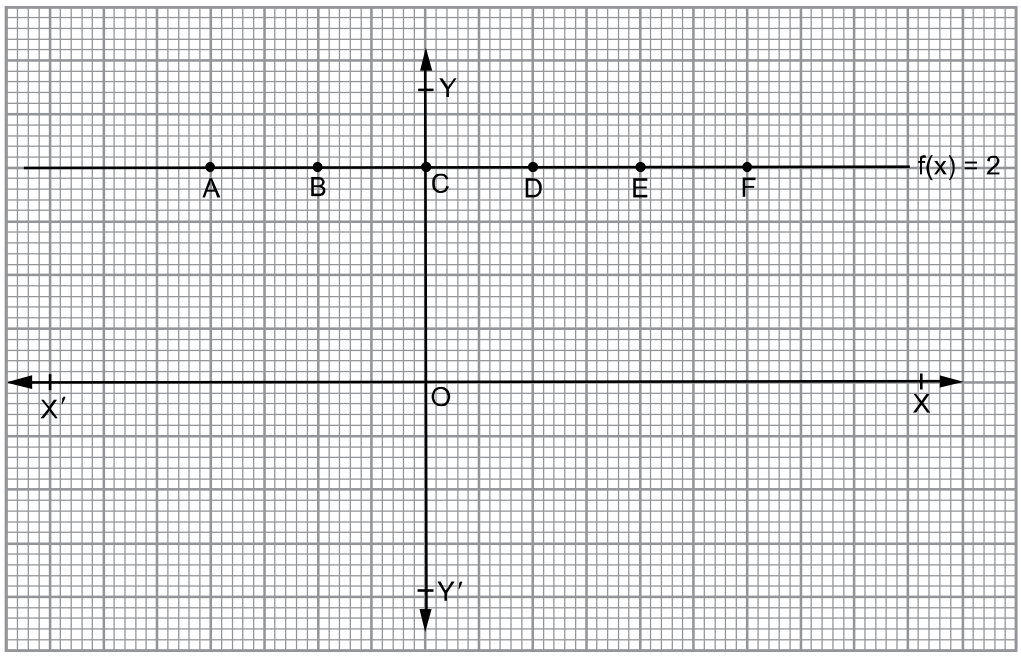
We may prepare the table given below.

x	–2	–1	0	1	2	3
$f(x) = 2$	2	2	2	2	2	2

On a graph paper, we draw $X'OX$ and YOY' as the x-axis and the y-axis respectively.

We take the scale: 10 small divisions = 1 unit.

On this graph paper, we plot the points $A(-2, 2)$, $B(-1, 2)$, $C(0, 2)$, $D(1, 2)$, $E(2, 2)$ and $F(3, 2)$. Join A, B, C, D, E and F successively to obtain the required graph line $ABCDEF$, whose equation is $y = 2$.



Graph of the function, $f(x) = 2$

(ii) Let $f(x) = 0$ for all $x \in R$. Then,

dom $(f) = R$ and range $(f) = \{0\}$.

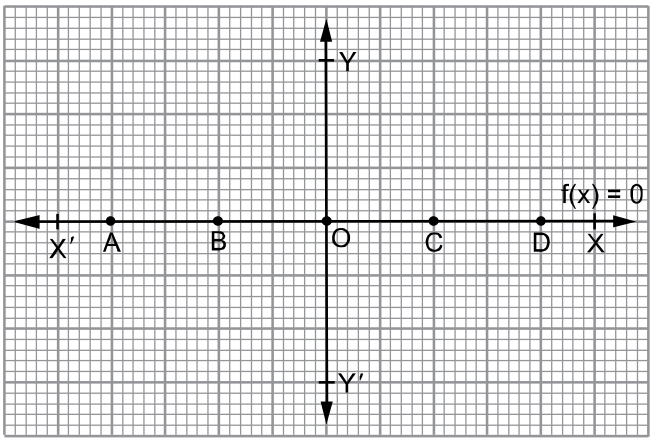
Now, we have:

x	–2	–1	0	1	2
$f(x) = 0$	0	0	0	0	0

On a graph paper, we draw the horizontal line $X'OX$ as the x-axis and the vertical line YOY' as the y-axis.

We take the scale: 10 small divisions = 1 unit.

Now, on this graph paper, we plot the points $A(-2, 0)$, $B(-1, 0)$, $O(0, 0)$, $C(1, 0)$ and $D(2, 0)$ and join them successively to get the graph line $ABOCD$, whose equation is $y = 0$.



Graph of the function, $f(x) = 0$

(iii) Let $f(x) = -2$ for all $x \in R$. Then,

dom $(f) = R$ and range $(f) = -2$.

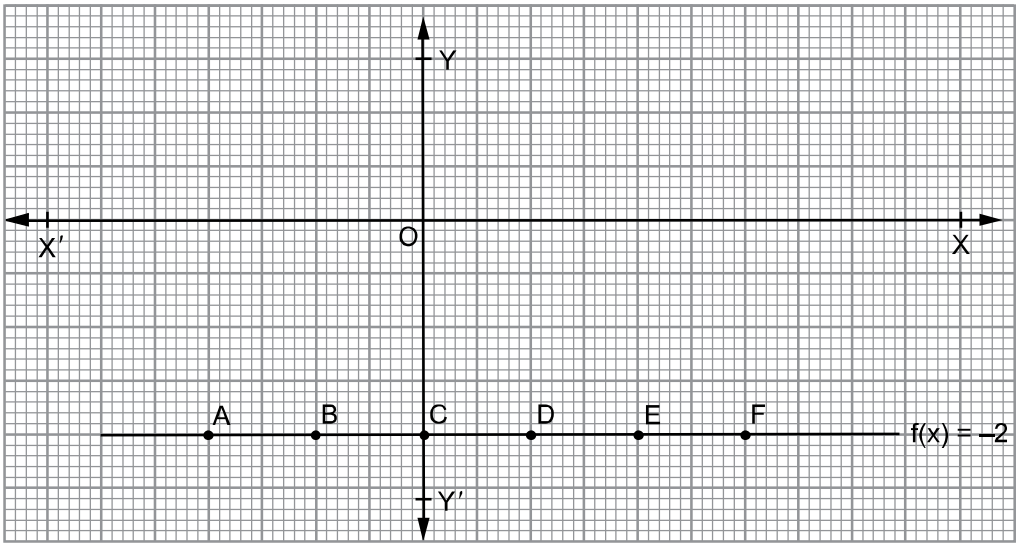
Now, we have:

x	–2	–1	0	1	2	3
$f(x) = -2$	–2	–2	–2	–2	–2	–2

On a graph paper, we draw the horizontal line $X'OX$ as the x-axis and the vertical line YOY' as the y-axis.

We take the scale: 10 small divisions = 1 unit.

On this graph paper, we plot the points $A(-2, -2)$, $B(-1, -2)$, $C(0, -2)$, $D(1, -2)$, $E(2, -2)$ and $F(3, -2)$ and join these points successively to get the required graph line $ABCDE$, as shown below. Its equation is $y = -2$.



Graph of the function, $f(x) = -2$

3. THE MODULUS FUNCTION

The function f, defined by $f(x) = |x| = \begin{cases} x, \text{ when } x \geq 0 \\ -x, \text{ when } x < 0 \end{cases}$ *is called the modulus function.*

We know that $|x| \geq 0$ for all $x \in R$.

$\therefore$ dom $(f) = R$ and range $(f) = [0, \infty)$.

EXAMPLE 3 *Draw the graph of the modulus function, defined by*

$$f : R \to R : f(x) = |x| = \begin{cases} x, \text{ when } x \geq 0 \\ -x, \text{ when } x < 0. \end{cases}$$

SOLUTION We have

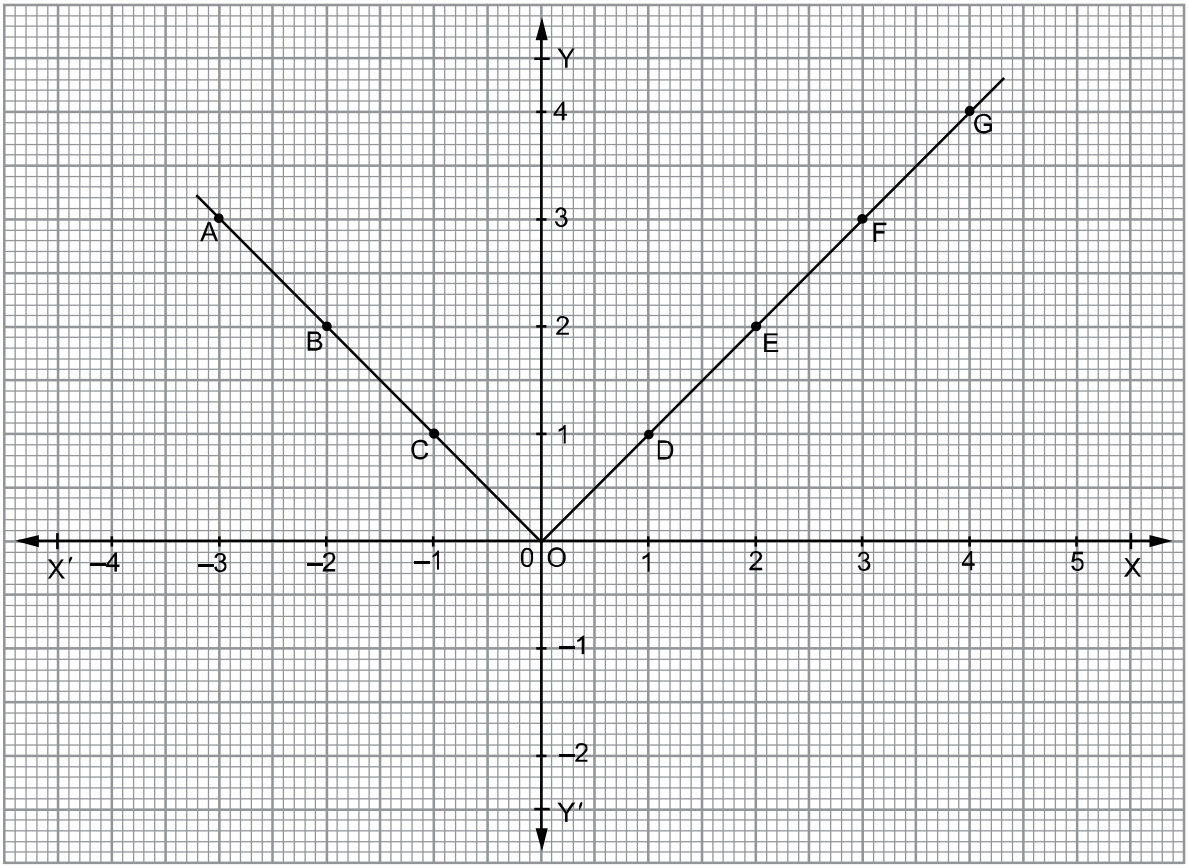
x	–3	–2	–1	0	1	2	3	4
$f(x) = \|x\|$	3	2	1	0	1	2	3	4

On a graph paper draw the horizontal line $X'OX$ as the *x*-axis and the vertical line YOY' as the *y*-axis.

Take the scale: 10 small divisions = 1 unit.

On this graph paper, plot the points *A*(–3, 3), *B*(–2, 2), *C*(–1, 1), *O*(0, 0), *D*(1, 1), *E*(2, 2), *F*(3, 3) and *G*(4, 4).

Join these points successively to obtain the graph lines *ABCO* and *ODEFG*, as shown below.



Graph of $f(x) = |x|$

4. THE GREATEST INTEGER FUNCTION, OR STEP FUNCTION

For any real number x, the symbol $[x]$ denotes the greatest integer less than or equal to x.

EXAMPLES (i) $[6.85] = 6$ (ii) $[8] = 8$ (iii) $[0.536] = 0$ (iv) $[-5.8] = -6$
(v) $[-5] = -5$

The function $f : R \to R : f(x) = [x]$ *for all* $x \in R$ is called the *greatest integer function, or step function.*

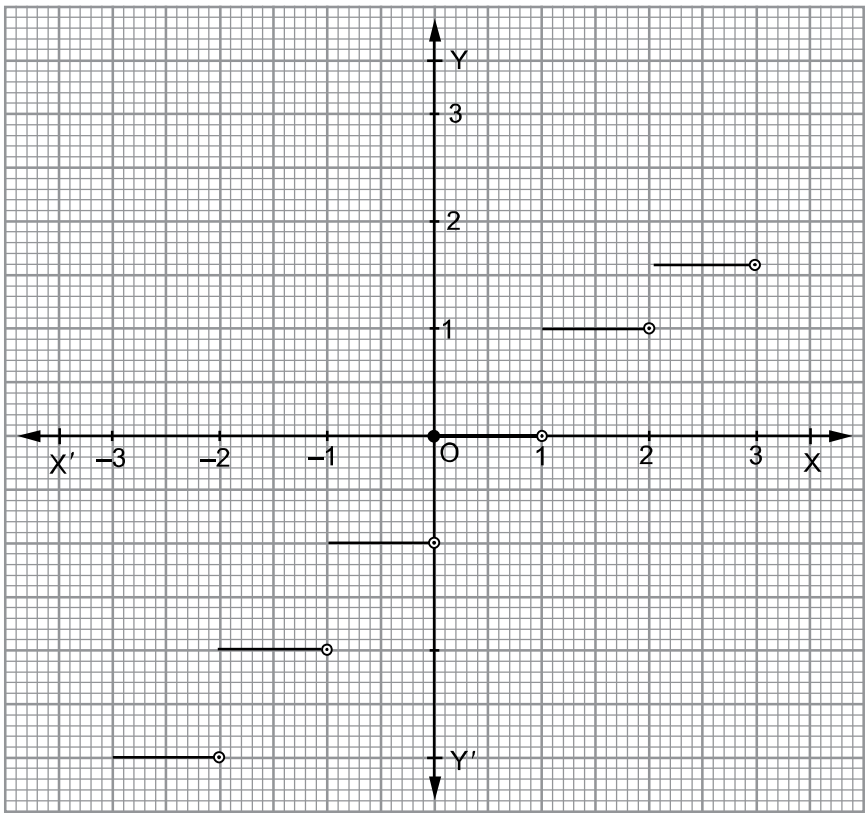
Clearly, dom $(f) = R$ and range $(f) = Z$.

EXAMPLE 4 *Draw the graph of the greatest integer function:*
$f : R \to R : f(x) = [x]$ *for all* $x \in R$.

SOLUTION Using the definition of $[x]$, we have

$[x] = -2$ for $-2 \le x < -1$
$[x] = -1$ for $-1 \le x < 0$
$[x] = 0$ for $0 \le x < 1$
$[x] = 1$ for $1 \le x < 2$
$[x] = 2$ for $2 \le x < 3$
$[x] = 3$ for $3 \le x < 4$.

Taking the scale: 10 small divisions = 1 unit, the graph of the function $f(x) = [x]$ may be drawn as shown below.



Graph of greatest integer function [*x*]

5. THE SMALLEST INTEGER FUNCTION

For any real number x, the smallest integer greater than or equal to x is denoted by $\lceil x \rceil$.

For example: $\lceil 3.8 \rceil = 4,\ \lceil 6 \rceil = 6,\ \lceil 0.76 \rceil = 1,\ \lceil -6.2 \rceil = -6$, etc.

The function $f : R \to R : f(x) = \lceil x \rceil$ *for all* $x \in R$ is called the *smallest integer function.*

Clearly, dom $(f) = R$, range $(f) = Z$.

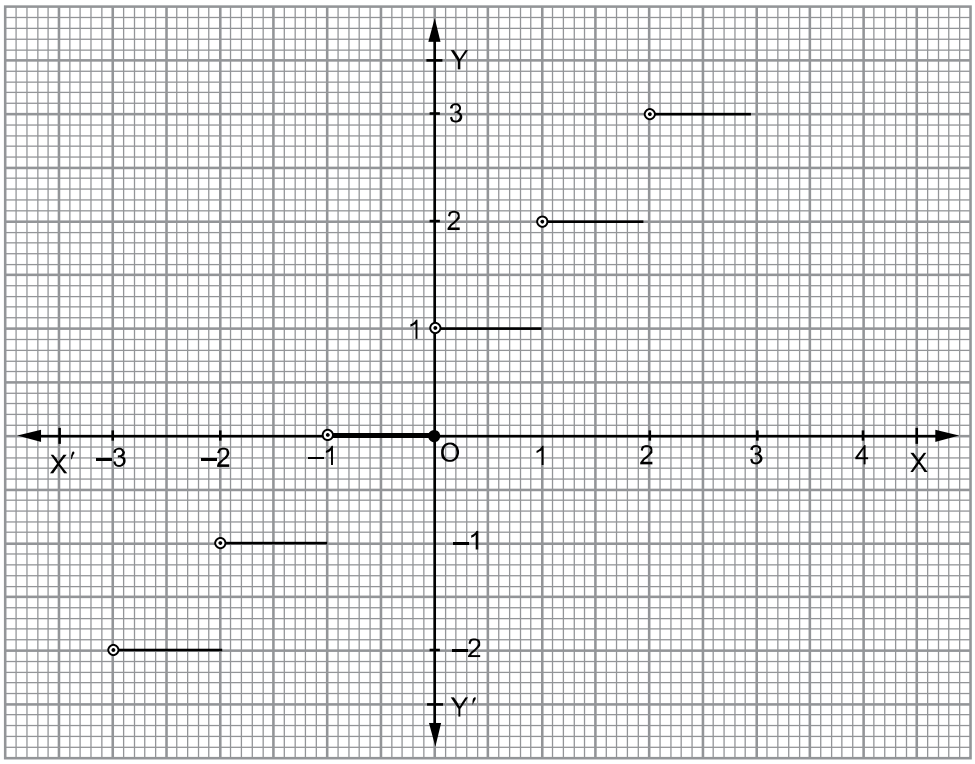
EXAMPLE 5 *Draw the graph of the smallest integer function* $f : R \to R : f(x) = \lceil x \rceil$ *for all* $x \in R$.

SOLUTION Using the definition of $\lceil x \rceil$, we have

x	$f(x) = \lceil x \rceil$
$-3 < x \leq -2$	–2
$-2 < x \leq -1$	–1
$-1 < x \leq 0$	0
$0 < x \leq 1$	1
$1 < x \leq 2$	2
$2 < x \leq 3$	3

SCALE: 10 small divisions = 1 unit.

Now, the graph may be drawn as shown below.



Graph of smallest integer function $[x]$

6. THE FRACTIONAL PART FUNCTION

Let $f : R \to R : f(x) = x - [x] = \{x\}$.

Here $f(x) = \{x\}$ denotes the fractional part of x.

For example: $\{2.35\} = 0.35, \{6\} = 0, \{-7\} = 0, \{-0.65\} = 0.35, \{-3.75\} = 0.25$, etc.

Clearly, dom $(f) = R$ and range $(f) = [0, 1)$.

EXAMPLE 6 *Draw the graph of the fractional part function:*

$$f : R \to R : f(x) = x - [x] = \{x\}$$

SOLUTION We have, $f(x) = x - [x] = \{x\}$.

$f(3) = 0, f(3.1) = 0.1, f(3.2) = 0.2, \ldots, f(3.9) = 0.9, f(4) = 0.$

$f(2) = 0, f(2.1) = 0.1, f(2.2) = 0.2, \ldots, f(2.9) = 0.9, f(3) = 0.$

$f(1) = 0, f(1.1) = 0.1, f(1.2) = 0.2, \ldots, f(1.9) = 0.9, f(2) = 0.$

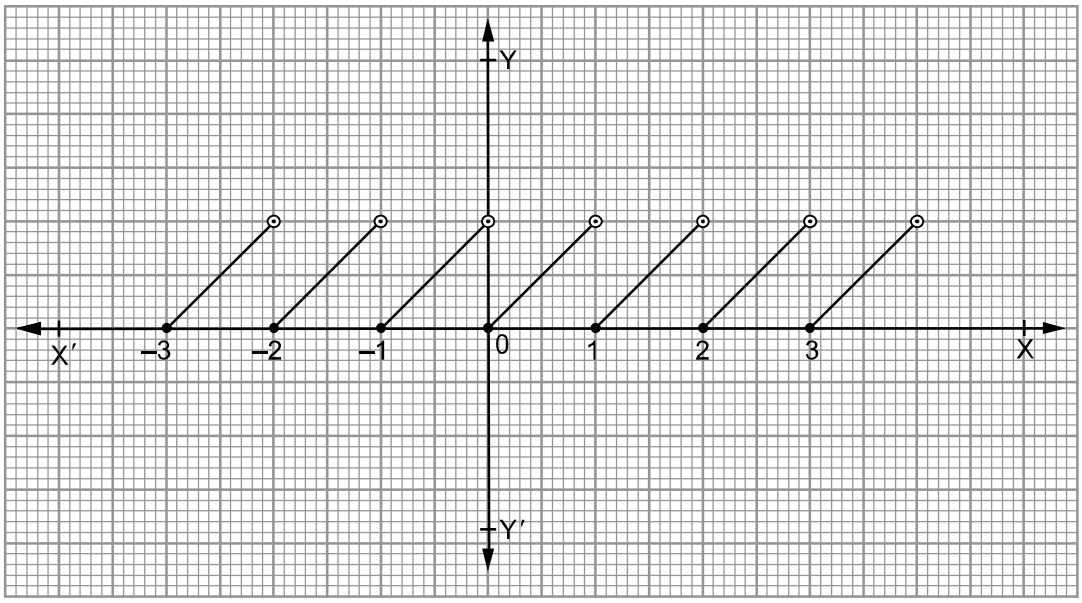
$f(0) = 0, f(0.1) = 0.1, f(0.2) = 0.2, \ldots, f(0.9) = 0.9, f(1) = 0.$

$f(-1) = 0, f(-0.9) = 0.1, f(-0.8) = 0.2, \ldots, f(-0.1) = 0.9, f(0) = 0.$

$f(-2) = 0, f(-1.9) = 0.1, f(-1.8) = 0.2, \ldots, f(-1.1) = 0.9, f(-1) = 0.$

$f(-3) = 0, f(-2.9) = 0.1, f(-2.8) = 0.2, \ldots, f(-2.1) = 0.9, f(-2) = 0.$

Plotting these points on a graph paper, we may get the graph as shown below.



Graph of fractional part function, $f(x) = \{x\}$

7. THE SIGNUM FUNCTION

The function $f : R \to R : f(x) = \begin{cases} \dfrac{x}{|x|}, \text{ when } x \neq 0 \\ 0, \text{ when } x = 0 \end{cases}$ *is called the* ***signum function.***

The above function may be simplified as under:

$$f(x) = \begin{cases} 1, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -1, \text{ if } x < 0. \end{cases}$$

Clearly, dom $(f) = R$ and range $(f) = \{-1, 0, 1\}$.

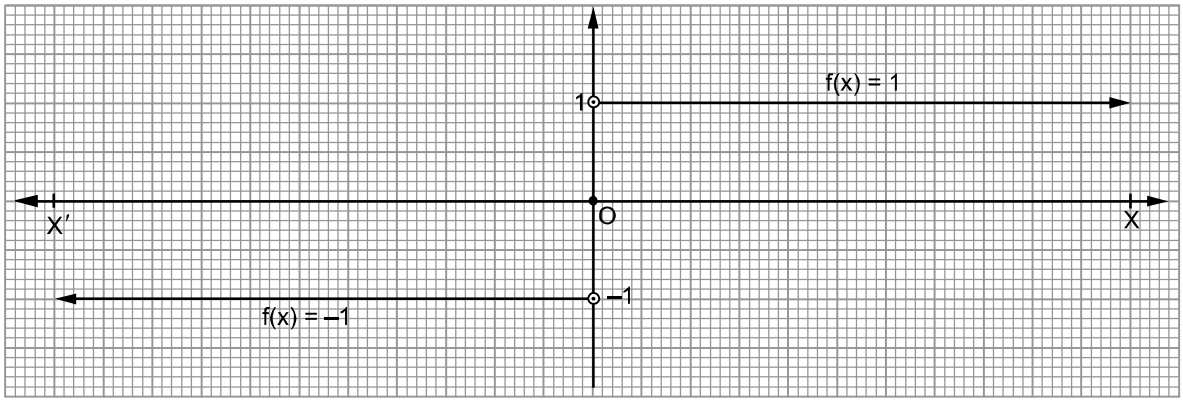
EXAMPLE 7 *Draw the graph of the signum function,* $f : R \to R$, *defined by*

$$f(x) = \begin{cases} \dfrac{x}{|x|}, \text{ when } x \neq 0 \\ 0, \text{ when } x = 0 \end{cases} \text{ or } f(x) = \begin{cases} 1, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -1, \text{ if } x < 0 \end{cases}$$

SOLUTION Clearly, we have

$$x < 0 \Rightarrow f(x) = -1$$
$$x = 0 \Rightarrow f(x) = 0$$
$$x > 0 \Rightarrow f(x) = 1.$$

We may now draw the graph as shown below.



Graph of signum function

8. POLYNOMIAL FUNCTION

A function of the form $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, where n is a non-negative integer and $a_0, a_1, a_2, \ldots, a_n \in R$, is called a *polynomial function*.

EXAMPLES (i) $f(x) = x^3 - 2x^2 + 5$ is a polynomial function.

(ii) $g(x) = 4x^2 - x + 2$ is a polynomial function.

(iii) $h(x) = 2x - 3\sqrt{x} + 5$ is not a polynomial function.

EXAMPLE 8 *Let* $f : R \to R : f(x) = x^2$ *for all* $x \in R$.

Find its domain and range.

Also, draw its graph.

SOLUTION Here, $f : R \to R : f(x) = x^2$ for all $x \in R$.

Dom $(f) = R$ and range $(f) = \{x \in R : x \geq 0\} = [0, \infty)$.

Now, we have:

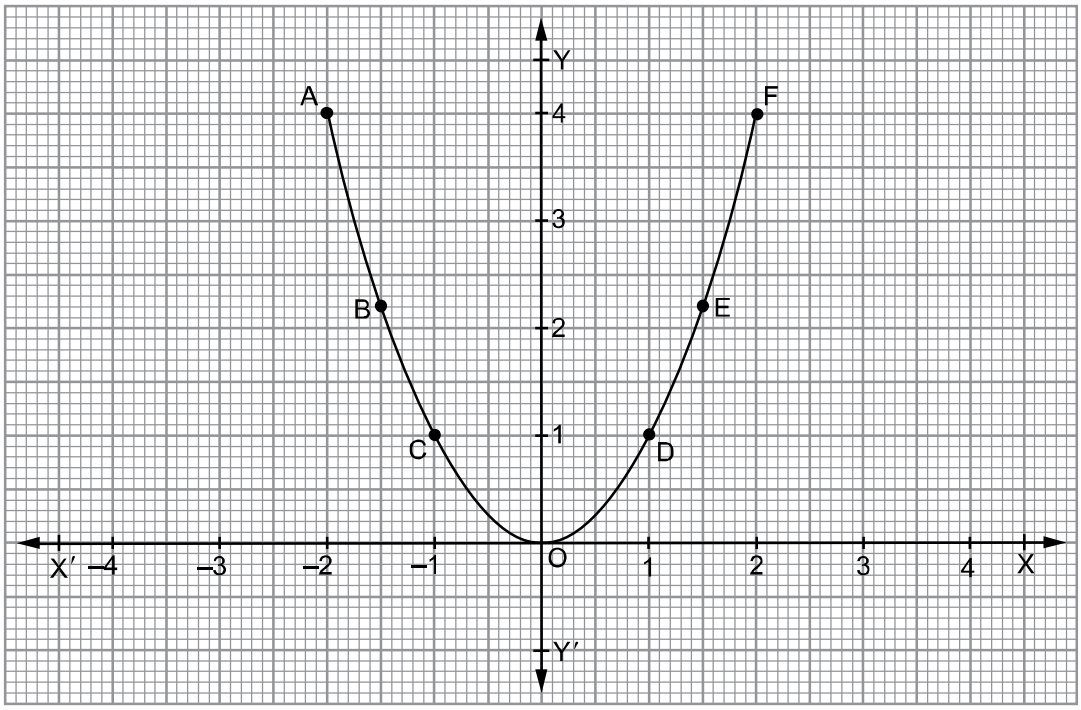
x	–2	–1.5	–1	0	1	1.5	2
$f(x) = x^2$	4	2.25	1	0	1	2.25	4

On a graph paper, we draw $X'OX$ and YOY' as the x-axis and the y-axis respectively.

Now, we plot the points $A(-2, 4)$, $B(-1.5, 2.25)$, $C(-1, 1)$, $O(0, 0)$, $D(1, 1)$, $E(1.5, 2.25)$ and $F(2, 4)$.

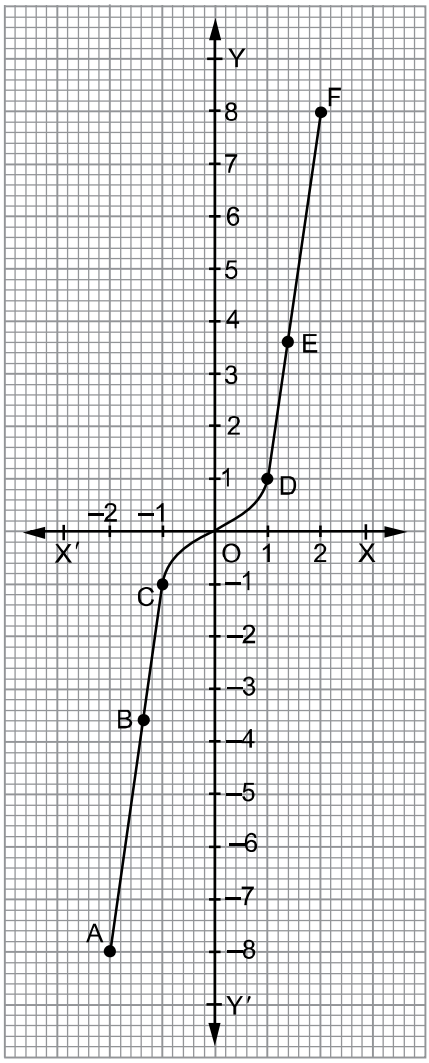
Join these points freehand successively to obtain the required curve.

Scale: 10 small divisions = 1 unit.



Graph of $f(x) = x^2$

EXAMPLE 9 *Let $f : R \to R : f(x) = x^3$ for all $x \in R$.*
Find its domain and range.
Also, draw its graph.



SOLUTION Let $f : R \to R : f(x) = x^3$ for all $x \in R$. Then,

dom $(f) = R$ and range $(f) = R$.

Now, we have:

x	–2	–1.5	–1	0	1	1.5	2
$f(x) = x^3$	–8	–3.375	–1	0	1	3.375	8

On a graph paper, we draw $X'OX$ and YOY' as the *x*-axis and the *y*-axis respectively.

We take the scale as

5 small divisions = 1 unit.

Now, we plot the points *A*(–2, –8), *B*(–1.5, –3.375), *C*(–1, –1), *O*(0, 0), *D*(1, 1), *E*(1.5, 3.375) and *F*(2, 8).

We join these points freehand successively to obtain the required curve.

9. RATIONAL FUNCTION

The functions of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x, defined in a domain, where $g(x) \neq 0$.

EXAMPLE 10 *Let* $f : (R - \{0\}) \to R : f(x) = \frac{1}{x}$ *for all values of* $x \in R - \{0\}$. *Find its domain and range. Also, draw its graph.*

SOLUTION Let $f : (R - \{0\}) \to R : f(x) = \frac{1}{x}$. Then,

$$\text{dom}\,(f) = R - \{0\} \text{ and } \text{range}\,(f) = R - \{0\}.$$

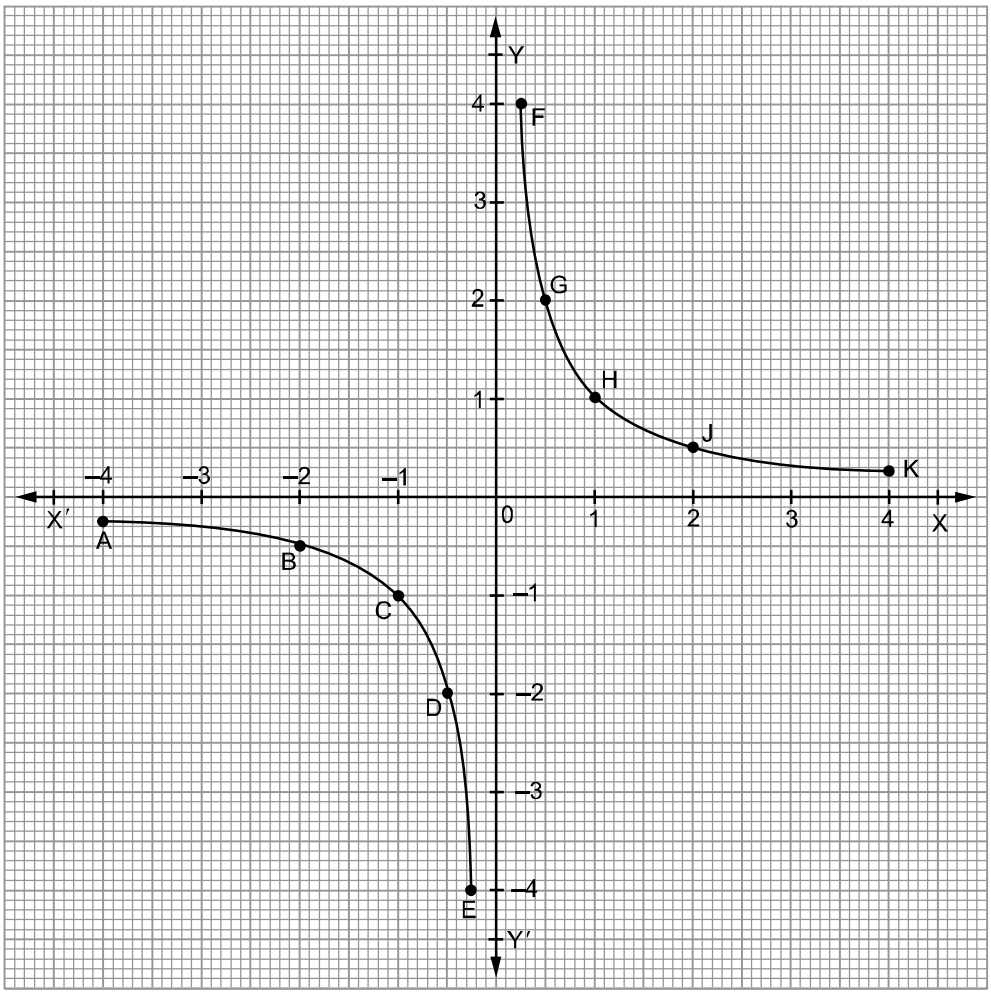
We have:

x	–4	–2	–1	–0.5	–0.25	0.25	0.5	1	2	4
$f(x) = \frac{1}{x}$	–0.25	–0.5	–1	–2	–4	4	2	1	0.5	0.25

On a graph paper, let $X'OX$ and YOY' be the x-axis and the y-axis respectively.

We take the scale: 10 small divisions = 1 unit.

Now, we plot the points $A(-4, -0.25)$, $B(-2, -0.5)$, $C(-1, -1)$, $D(-0.5, -2)$, $E(-0.25, -4)$ and $F(0.25, 4)$, $G(0.5, 2)$, $H(1, 1)$, $J(2, 0.5)$ and $K(4, 0.25)$.



Graph of the function, $f(x) = \frac{1}{x}$

We join *A, B, C, D* and *E* freehand successively to obtain a curve. Also, we join *F, G, H, J* and *K* freehand successively to obtain another curve, as shown above.

EXERCISE 3D

1. Consider the real function $f : R \to R : f(x) = x + 5$ for all $x \in R$.
Find its domain and range.
Draw the graph of this function.

2. Consider the function $f : R \to R$, defined by

$$f(x) = \begin{cases} 1 - x, \text{ when } x < 0 \\ 1, \text{ when } x = 0 \\ x + 1, \text{ when } x > 0. \end{cases}$$

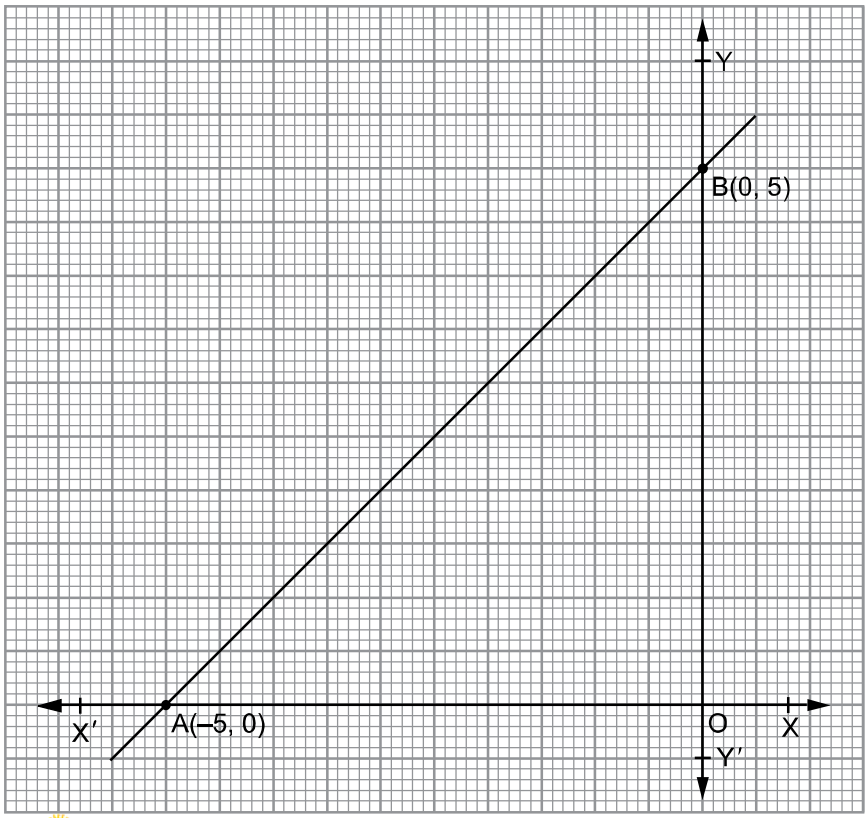
Write its domain and range.
Also, draw the graph of $f(x)$.

3. Find the domain and the range of the square root function, $f : R^+ \cup \{0\} \to R : f(x) = \sqrt{x}$ for all non-negative real numbers.
Also, draw its graph.

4. Find the domain and the range of the cube root function, $f : R \to R : f(x) = x^{1/3}$ for all $x \in R$.
Also, draw its graph.

HINTS TO SOME SELECTED QUESTIONS

1. Clearly, dom $(f) = R$ and range $(f) = R$.



Now, $y = x + 5$.

Putting $x = 0$, we get $y = 5$.

Putting $y = 0$, we get $x = -5$.

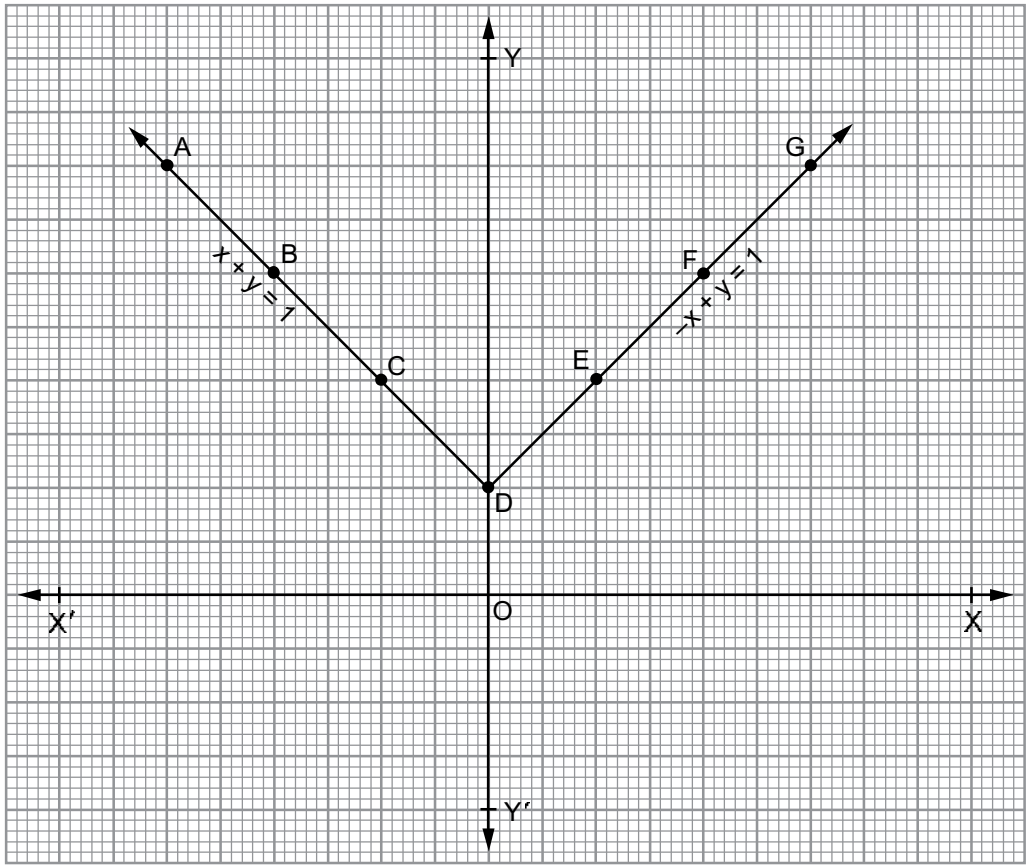
On the graph paper, plot the points $A(-5, 0)$ and $B(0, 5)$.

Join AB to get the required graph.

2. $x < 0 \Rightarrow y = 1 - x$.

$x = 1 \Rightarrow y = 1$.

$x > 1 \Rightarrow y = x + 1$.



(i) For $x < 0$, we have $y = 1 - x$.

x	–3	–2	–1
y	4	3	2

(ii) For $x = 0$, we have $y = 1$.

x	0
y	1

$[\because\ x = 0 \Rightarrow y = 1]$

(iii) For $x > 0$, we have $y = x + 1$.

x	1	2	3
y	2	3	4

On a graph paper, draw horizontal line $X'OX$ and the vertical line YOY'. Now, plot the points $A(-3, 4)$, $B(-2, 3)$, $C(-1, 2)$, $D(0, 1)$, $E(1, 2)$, $F(2, 3)$ and $G(3, 4)$.

Join $ABCD$ and $DEFG$ to get the required graph.

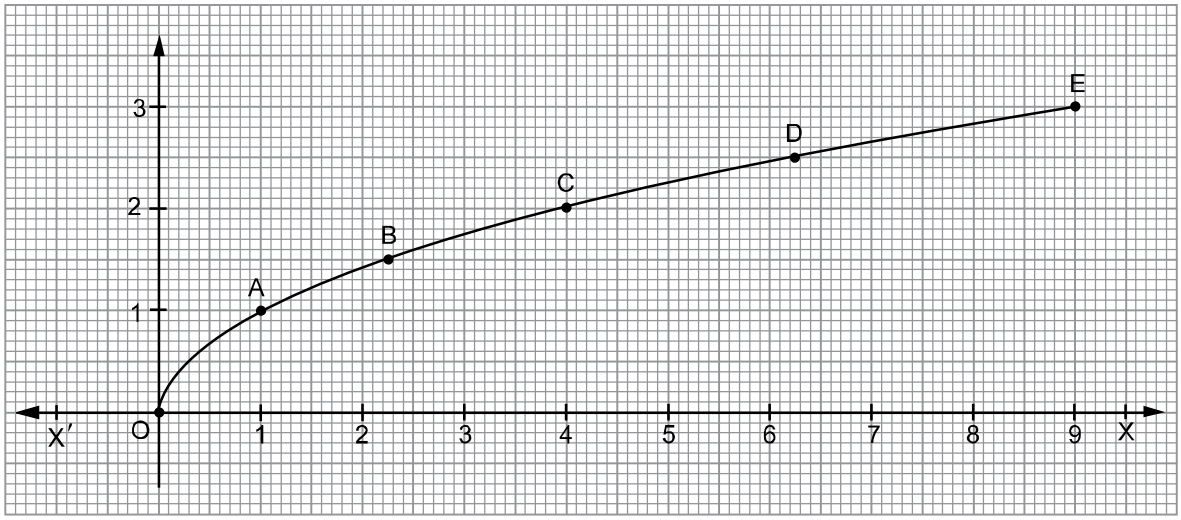
3. Let $f : R^+ \cup \{0\} \to R : f(x) = \sqrt{x}$. Then,

dom $(f) = [0, \infty)$ and range $(f) = [0, \infty)$.

We have

$\sqrt{0} = 0$, $\sqrt{1} = 1$, $\sqrt{2.25} = 1.5$, $\sqrt{4} = 2$, $\sqrt{6.25} = 2.5$ and $\sqrt{9} = 3$.

On a graph paper, we plot the points $O(0, 0)$, $A(1, 1)$, $B(2.25, 1.5)$, $C(4, 2)$, $D(6.25, 2.5)$, and $E(9, 3)$. Join these points freehand successively to get the required curve.



Graph of the curve, $f(x) = \sqrt{x}$

4. Let $f : R \to R : f(x) = x^{1/3}$ for all $x \in R$.

Clearly, dom $(f) = R$ and range $(f) = R$.

We have

$\sqrt[3]{8} = 2$, $\sqrt[3]{5} = 1.7$, $\sqrt[3]{3} = 1.4$, $\sqrt[3]{1} = 1$, $\sqrt[3]{0} = 0$, $\sqrt[3]{-1} = -1$, $\sqrt[3]{-3} = -1.4$, $\sqrt[3]{-5} = -1.7$, and $\sqrt[3]{-8} = -2$.

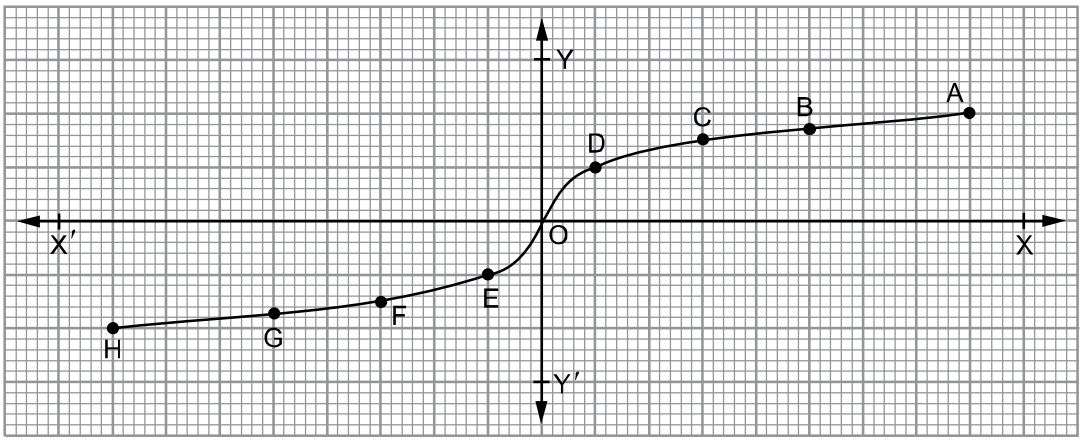
On a graph paper, take the horizontal line $X'OX$ as the x-axis and the vertical line YOY' as the y-axis.

Take the scale: 5 small divisions = 1 unit.

Now, plot the points $A(8, 2)$, $B(5, 1.7)$, $C(3, 1.4)$, $D(1, 1)$, $O(0, 0)$, $E(-1, -1)$, $F(-3, -1.4)$, $G(-5, -1.7)$ and $H(-8, -2)$.

Join $ABCDOEFGH$ freehand to get the required curve.

Remember: $\sqrt[3]{3} = 1.4$ and $\sqrt[3]{5} = 1.7$.



OPERATIONS ON FUNCTIONS

(i) SUM OF TWO REAL FUNCTIONS

Let $f: D_1 \to R$ and $g: D_2 \to R$, where $D_1 \subseteq R$ and $D_2 \subseteq R$. Then,

$(f+g): (D_1 \cap D_2) \to R: (f+g)(x) = f(x) + g(x)$ for all $x \in (D_1 \cap D_2)$.

(ii) DIFFERENCE OF TWO REAL FUNCTIONS

Let $f: D_1 \to R$ and $g: D_2 \to R$, where $D_1 \subseteq R$ and $D_2 \subseteq R$. Then,

$(f-g): (D_1 \cap D_2) \to R: (f-g)(x) = f(x) - g(x)$ for all $x \in (D_1 \cap D_2)$.

(iii) MULTIPLICATION OF TWO REAL FUNCTIONS

Let $f: D_1 \to R$ and $g: D_2 \to R$, where $D_1 \subseteq R$ and $D_2 \subseteq R$. Then,

$(fg): (D_1 \cap D_2) \to R: (fg)(x) = f(x) \cdot g(x)$ for all $x \in (D_1 \cap D_2)$.

NOTE This is known as pointwise multiplication of functions.

(iv) QUOTIENT OF TWO REAL FUNCTIONS

Let $f: D_1 \to R$ and $g: D_2 \to R$, where $D_1 \subseteq R$ and $D_2 \subseteq R$.

Let $D = (D_1 \cap D_2) - \{x: g(x) = 0\}$. Then,

$$\left(\frac{f}{g}\right): D \to R: \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ for all } x \in D.$$

(v) RECIPROCAL OF A FUNCTION

Let $f: D_1 \to R$ and let $D = D_1 - \{x: f(x) = 0\}$.

Then, the reciprocal of f is the function $\frac{1}{f}$, defined by

$$\frac{1}{f}: D \to R: \left(\frac{1}{f}\right)(x) = \frac{1}{f(x)} \text{ for all } x \in D.$$

(vi) SCALAR MULTIPLE OF A FUNCTION

Let $f: D \to R$ and let α be a scalar (a real number). Then,

$(\alpha f): D \to R: (\alpha f)(x) = \alpha \cdot f(x)$ for all $x \in D$.

SOLVED EXAMPLES

EXAMPLE 1 *Let* $f: R \to R: f(x) = x^2$ *and* $g: R \to R: g(x) = 2x + 1$.

Find (i) $(f+g)(x)$ *(ii)* $(f-g)(x)$ *(iii)* $(fg)(x)$ *(iv)* $\left(\frac{f}{g}\right)(x)$

SOLUTION Here, dom $(f) = R$ and dom $(g) = R$.

$\therefore$ dom $(f) \cap$ dom $(g) = (R \cap R) = R$.

(i) $(f+g): R \to R$ is given by

$(f+g)(x) = f(x) + g(x) = x^2 + (2x+1) = (x+1)^2$.

(ii) $(f-g): R \to R$ is given by

$(f-g)(x) = f(x) - g(x) = x^2 - (2x+1) = (x^2 - 2x - 1)$.

(iii) $(fg): R \to R$ is given by

$(fg)(x) = f(x) \cdot g(x) = x^2 \cdot (2x+1) = (2x^3 + x^2)$.

(iv) $\{x : g(x) = 0\} = \{x : 2x+1 = 0\} = \left\{\frac{-1}{2}\right\}$.

$\therefore \quad \text{dom}\left(\frac{f}{g}\right) = R \cap R - \left\{\frac{-1}{2}\right\} = R - \left\{\frac{-1}{2}\right\}$.

The function $\frac{f}{g}: R - \left\{\frac{-1}{2}\right\} \to R$ is given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2}{2x+1}, x \neq \frac{-1}{2}.$$

EXAMPLE 2 *Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined over the set of non-negative real numbers. Find:*

(i) $(f+g)(x)$ *(ii)* $(f-g)(x)$ *(iii)* $(fg)(x)$ *(iv)* $\frac{f}{g}(x)$

SOLUTION Here $f : [0, \infty) \to R : f(x) = \sqrt{x}$ and $g : [0, \infty) \to R : g(x) = x$.

$\therefore \quad \text{dom}\,(f) = [0, \infty)$ and $\text{dom}\,(g) = [0, \infty)$.

So, $\text{dom}\,(f) \cap \text{dom}\,(g) = [0, \infty) \cap [0, \infty) = [0, \infty)$.

(i) $(f+g): [0, \infty) \to R$ is given by

$(f+g)(x) = f(x) + g(x) = (\sqrt{x} + x)$.

(ii) $(f-g): [0, \infty) \to R$ is given by

$(f-g)(x) = f(x) - g(x) = (\sqrt{x} - x)$.

(iii) $(fg): [0, \infty) \to R$ is given by

$(fg)(x) = f(x) \cdot g(x) = (\sqrt{x} \times x) = x^{3/2}$.

(iv) $\{x : g(x) = 0\} = \{0\}$.

$\therefore \quad \text{dom}\left(\frac{f}{g}\right) = \text{dom}\,(f) \cap \text{dom}\,(g) - \{x : g(x) = 0\}$

$= [0, \infty) \cap [0, \infty) - \{0\} = (0, \infty)$.

So, $\frac{f}{g}: (0, \infty) \to R$ is given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}, x \neq 0.$$

EXAMPLE 3 *Let f and g be real functions, defined by $f(x) = \frac{1}{(x+4)}$ and $g(x) = (x+4)^3$.*

Find (i) $(f+g)(x)$ *(ii)* $(f-g)(x)$ *(iii)* $(fg)(x)$ *(iv)* $\left(\frac{f}{g}\right)(x)$ *(v)* $\left(\frac{1}{f}\right)(x)$

SOLUTION Clearly, $f(x) = \frac{1}{(x+4)}$ is defined for all real values of x except that at which $x + 4 = 0$, i.e., $x = -4$.

$\therefore \quad \text{dom}\,(f) = R - \{-4\}$.

And, $g(x) = (x+4)^3$ is defined for all $x \in R$. So, dom $(g) = R$.

$\therefore$ dom $(f) \cap$ dom $(g) = R - \{-4\} \cap R = R - \{-4\}$.

(i) $(f+g) : R - \{-4\} \to R$ is given by

$$(f+g)(x) = f(x) + g(x) = \frac{1}{(x+4)} + (x+4)^3 = \frac{1+(x+4)^4}{(x+4)}.$$

(ii) $(f-g) : R - \{-4\} \to R$ is given by

$$(f-g)(x) = f(x) - g(x) = \frac{1}{(x+4)} - (x+4)^3 = \frac{1-(x+4)^4}{(x+4)}.$$

(iii) $(fg) : R - \{-4\} \to R$ is given by

$$(fg)(x) = f(x) \cdot g(x) = \frac{1}{(x+4)} \times (x+4)^3 = (x+4)^2.$$

(iv) $\{x : g(x) = 0\} = \{x : (x+4)^3 = 0\} = \{x : x+4 = 0\} = \{-4\}$.

$\therefore$ dom $\left(\frac{f}{g}\right)$ = dom $(f) \cap$ dom $(g) - \{x : g(x) = 0\} = R - \{-4\}$.

$\left(\frac{f}{g}\right) : R - \{-4\} \to R$ is given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\frac{1}{(x+4)}}{(x+4)^3} = \frac{1}{(x+4)^4},\ x \neq -4.$$

(v) Clearly, $f(x) \neq 0$ for any $x \in R - \{-4\}$.

$\therefore$ $\frac{1}{f} : R - \{-4\} \to R$ is given by

$$\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)} = \frac{1}{\frac{1}{(x+4)}} = (x+4).$$

EXAMPLE 4 *Let f and g be real functions defined by* $f(x) = \sqrt{x-1}$ *and* $g(x) = \sqrt{x+1}$.

Find (i) $(f+g)(x)$ *(ii)* $(f-g)(x)$ *(iii)* $(fg)(x)$ *(iv)* $\left(\frac{f}{g}\right)(x)$.

SOLUTION Clearly, $f(x) = \sqrt{x-1}$ is defined for all real values of x for which $x - 1 \geq 0$, i.e., $x \geq 1$. So, dom $(f) = [1, \infty)$.

Also, $g(x) = \sqrt{x+1}$ is defined for all real values of x for which $x + 1 \geq 0$, i.e., $x \geq -1$. So, dom $(g) = [-1, \infty)$.

$\therefore$ dom $(f) \cap$ dom $(g) = \{1, \infty) \cap [-1, \infty) = [1, \infty)$.

(i) $(f+g) : [1, \infty) \to R$ is given by

$$(f+g)(x) = f(x) + g(x) = (\sqrt{x-1} + \sqrt{x+1}).$$

(ii) $(f-g) : [1, \infty) \to R$ is given by

$$(f-g)(x) = f(x) - g(x) = (\sqrt{x-1} - \sqrt{x+1}).$$

(iii) $(fg) : [1, \infty) \to R$ is given by

$$(fg)(x) = f(x) \cdot g(x) = \sqrt{x-1} \times \sqrt{x+1} = \sqrt{x^2-1}.$$

(iv) $\{x : g(x) = 0\} = \{x : \sqrt{x+1} = 0\} = \{x : x+1 = 0\} = \{-1\}$.

$\therefore$ dom $(f) \cap$ dom $(g) - \{x : g(x) = 0\}$

$= [1, \infty) \cap [-1, \infty) - \{-1\} = [1, \infty)$.

$\therefore$ $\frac{f}{g} \to [1, \infty) \to R$ is given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x-1}}{\sqrt{x+1}}.$$

EXAMPLE 5 *Let f and g be real functions, defined by* $f(x) = \sqrt{x+2}$ *and* $g(x) = \sqrt{4-x^2}$. *Find (i)* $(f+g)(x)$ *(ii)* $(f-g)(x)$ *(iii)* $(fg)(x)$ *(iv)* $(ff)(x)$ *(v)* $(gg)(x)$ *(vi)* $\left(\frac{f}{g}\right)(x)$.

SOLUTION Clearly, $f(x) = \sqrt{x+2}$ is defined for all $x \in R$ such that

$x + 2 \geq 0$, i.e., $x \geq -2$.

$\therefore$ dom $(f) = [-2, \infty)$.

Again, $g(x) = \sqrt{4-x^2}$ is defined for all $x \in R$ such that

$4 - x^2 \geq 0$.

But, $4 - x^2 \geq 0 \Rightarrow x^2 - 4 \leq 0 \Rightarrow (x+2)(x-2) \leq 0 \Rightarrow x \in [-2, 2]$.

$\therefore$ dom $(g) = [-2, 2]$.

$\therefore$ dom $(f) \cap$ dom $(g) = [-2, \infty) \cap [-2, 2] = [-2, 2]$.

(i) $(f+g) : [-2, 2] \to R$ is given by

$$(f+g)(x) = f(x) + g(x) = \sqrt{x+2} + \sqrt{4-x^2}.$$

(ii) $(f-g) : [-2, 2] \to R$ is given by

$$(f-g)(x) = f(x) - g(x) = \sqrt{x+2} - \sqrt{4-x^2}.$$

(iii) $(fg) : [-2, 2] \to R$ is given by

$$(fg)(x) = f(x) \cdot g(x) = (\sqrt{x+2})(\sqrt{4-x^2})$$

$$= \sqrt{(x+2)^2(2-x)} = (x+2)\sqrt{(2-x)}.$$

(iv) $(ff) : [-2, 2] \to R$ is given by

$$(ff)(x) = f(x) \cdot f(x) = (\sqrt{x+2})(\sqrt{x+2}) = (x+2).$$

(v) $(gg) : [-2, 2] \to R$ is given by

$$(gg)(x) = g(x) \cdot g(x) = (\sqrt{4-x^2})(\sqrt{4-x^2}) = (4-x^2).$$

(vi) $\{x : g(x) = 0\} = \{x : 4 - x^2 = 0] = \{x : (2-x)(2+x) = 0\} = \{-2, 2\}$.

$\therefore$ dom $\left(\frac{f}{g}\right)$ = dom $(f) \cap$ dom $\{g\} - \{x : g(x) = 0\}$

$= [-2, 2] - \{-2, 2\} = (-2, 2)$.

$\therefore$ $\frac{f}{g} : (-2, 2) \to R$ is given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x+2}}{\sqrt{4-x^2}} = \frac{\sqrt{2+x}}{(\sqrt{2+x})(\sqrt{2-x})} = \frac{1}{(\sqrt{2-x})}.$$

EXAMPLE 6 *Let f be the exponential function and g be the logarithmic function. Then, find:*

(i) $(f+g)(1)$ *(ii)* $(fg)(1)$ *(iii)* $(4f)(1)$ *(iv)* $(3g)(1)$

SOLUTION Let $f : R \to R : f(x) = e^x$ and $g : R^+ \to R : g(x) = \log_e x$.

Then, dom $(f) \cap$ dom $(g) = R \cap R^+ = R^+$.

(i) $(f+g) : R^+ \to R$ is given by

$$(f+g)(x) = f(x) + g(x) = (e^x + \log_e x).$$

$$\therefore \quad (f+g)(1) = (e^1 + \log_e 1) = (e+0) = e.$$

(ii) $(fg) : R^+ \to R$ is given by

$$(fg)(x) = f(x) \cdot g(x) = e^x(\log_e x).$$

$$\therefore \quad (fg)(1) = e^1(\log_e 1) = (e \times 0) = 0.$$

(iii) $(4f) : R^+ \to R$ is given by

$$(4f)(x) = 4 \times f(x) = 4e^x.$$

$$\therefore \quad (4f)(1) = (4 \times e^1) = 4e.$$

(iv) $(3g) : R^+ \to R$ is given by

$$(3g)(x) = 3 \times g(x) = 3 \times (\log_e x)$$

$$\therefore \quad (3g)(1) = 3 \times g(1) = 3 \times (\log_e 1) = (3 \times 0) = 0.$$

EXAMPLE 7 *If $f(x) = \log_e(1-x)$ and $g(x) = [x]$ then find:*

(i) $(f+g)(x)$ *(ii)* $(fg)(x)$ *(iii)* $\left(\frac{f}{g}\right)(x)$ *(iv)* $\left(\frac{g}{f}\right)(x)$.

Also find $(f+g)(-1), (fg)(0), \left(\frac{f}{g}\right)(-1), \left(\frac{g}{f}\right)\left(\frac{1}{2}\right)$.

SOLUTION Clearly, $\log_e(1-x)$ is defined only when $1-x > 0$, i.e., $x < 1$.

$$\therefore \quad \text{dom } (f) = (-\infty, 1).$$

Also, dom $(g) = R$.

$$\therefore \quad \text{dom } (f) \cap \text{dom } (g) = (-\infty, 1) \cap R = (-\infty, 1).$$

(i) $(f+g) : (-\infty, 1) \to R$ is given by

$$(f+g)(x) = f(x) + g(x) = \log_e(1-x) + [x]$$

(ii) $(fg) : (-\infty, 1) \to R$ is given by

$$(fg)(x) = f(x) \times g(x) = \{\log_e(1-x)\} \times [x].$$

(iii) $\{x : g(x) = 0\} = \{x : [x] = 0\} = [0, 1)$.

$$\therefore \quad \text{dom}\left(\frac{f}{g}\right) = \text{dom } (f) \cap \text{dom } (g) - \{x : g(x) = 0\}$$

$$= (-\infty, 1) \cap R - [0, 1) = (-\infty, 0).$$

$\therefore \quad \frac{f}{g} : (-\infty, 0) \to R$ is given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\log_e(1-x)}{[x]}.$$

(iv) $\{x : f(x) = 0\} = \{x : \log_e(1-x) = 0\} = \{0\}.$

$$\therefore \quad \text{dom}\left(\frac{g}{f}\right) = \text{dom}\,(g) \cap \text{dom}\,(f) - \{x : f(x) = 0\}$$
$$= R \cap (-\infty, 1) - \{0\} = (-\infty, 0) \cup (0, 1).$$

$\therefore \quad \frac{g}{f} : (-\infty, 0) \cup (0, 1) \to R$ is given by

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{[x]}{\log_e(1-x)}.$$

Now, we have:

$(f+g)(-1) = f(-1) + g(-1) = [-1] + \log_e(1+1) = (\log_e 2) - 1.$

$(fg)(0) = f(0) \times g(0) = \log_e(1-0) \times [0] = (\log_e 1 \times 0) = (0 \times 0) = 0.$

$$\left(\frac{f}{g}\right)(-1) = \frac{f(-1)}{g(-1)} = \frac{[-1]}{\log_e(1+1)} = \frac{-1}{\log_e 2}.$$

$$\left(\frac{g}{f}\right)\left(\frac{1}{2}\right) = \frac{g\left(\frac{1}{2}\right)}{f\left(\frac{1}{2}\right)} = \frac{\left[\frac{1}{2}\right]}{\log_e\left(1-\frac{1}{2}\right)} = \frac{[0.5]}{\log_e\left(\frac{1}{2}\right)} = 0.$$

EXAMPLE 8 *Find the sum and the difference of the identity function and the modulus function.*

SOLUTION Let $f : R \to R : f(x) = x$ be the identity function.

And, let $g : R \to R : g(x) = |x|$ be the modulus function.

Then, dom $(f) = R$ and dom $(g) = R$.

$\therefore \quad$ dom $(f) \cap$ dom $(g) = R \cap R = R.$

(i) dom $(f+g) =$ dom $(f) \cap$ dom $(g) = R.$

Now, $(f+g) : R \to R$ is given by

$$(f+g)(x) = f(x) + g(x)$$
$$= x + |x| = x + \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$$
$$= \begin{cases} x + x, & \text{when } x \geq 0 \\ x - x, & \text{when } x < 0 \end{cases} = \begin{cases} 2x, & \text{when } x \geq 0 \\ 0, & \text{when } x < 0. \end{cases}$$

Hence, $(f+g)(x) = \begin{cases} 2x, & \text{when } x \geq 0 \\ 0, & \text{when } x < 0. \end{cases}$

(ii) dom $(f-g) =$ dom $(f) \cap$ dom $(g) = R.$

$\therefore \quad (f-g)(x) = f(x) - g(x)$

$$= x - |x| = x - \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$$

$$= \begin{cases} x - x, \text{ when } x \geq 0 \\ x + x, \text{ when } x < 0 \end{cases} = \begin{cases} 0, \text{ when } x \geq 0 \\ 2x, \text{ when } x < 0 \end{cases}$$

$$\therefore \quad (f-g)(x) = \begin{cases} 0, \text{ when } x \geq 0 \\ 2x, \text{ when } x < 0. \end{cases}$$

EXAMPLE 9 *Find the sum and the difference of the identity function and the reciprocal function.*

SOLUTION Let $f : R \to R : f(x) = x$ and $g : R - \{0\} \to R : g(x) = \frac{1}{x}$ be the identity function and the reciprocal function respectively.

Then, $\text{dom}(f) \cap \text{dom}(g) = R \cap R - \{0\} = R - \{0\}$.

$$\therefore \quad (f+g) : R - \{0\} \to R : (f+g)(x) = f(x) + g(x) = \left(x + \frac{1}{x}\right).$$

Hence, $(f+g)(x) = \left(x + \frac{1}{x}\right)$ for all $x \in R - \{0\}$.

And, $(f-g) : R - \{0\} \to R : (f-g)(x) = f(x) - g(x) = \left(x - \frac{1}{x}\right).$

Hence, $(f-g)(x) = \left(x - \frac{1}{x}\right)$ for all $x \in R - \{0\}$.

EXAMPLE 10 *Find the product of the identity function by the modulus function.*

SOLUTION Let $f : R \to R : f(x) = x$ and $g : R \to R : g(x) = |x|$ be the identity function and the modulus function respectively.

Then, $\text{dom}(fg) = \text{dom}(f) \cap \text{dom}(g) = R \cap R = R$.

$\therefore \quad (fg) : R \to R : (fg)(x) = f(x) \cdot g(x)$.

Now, $(fg)(x) = f(x) \cdot g(x) = x \cdot |x|$

$$= x \cdot \begin{cases} x, \text{ when } x \geq 0 \\ -x, \text{ when } x < 0 \end{cases} = \begin{cases} x^2, \text{ when } x \geq 0 \\ -x^2, \text{ when } x < 0. \end{cases}$$

Hence, $(fg)(x) = \begin{cases} x^2, \text{ when } x \geq 0 \\ -x^2, \text{ when } x < 0. \end{cases}$

EXAMPLE 11 *Find the product of the identity function by the reciprocal function.*

SOLUTION Let $f : R \to R : f(x) = x$ and $g : R - \{0\} \to R : g(x) = \frac{1}{x}$ be the identity function and the reciprocal function respectively.

Then, $\text{dom}(fg) = \text{dom}(f) \cap \text{dom}(g) = R \cap R - \{0\} = R - \{0\}$.

$$\therefore \quad (fg) : R - \{0\} \to R : (fg)(x) = f(x) \cdot g(x) = \left(x \times \frac{1}{x}\right) = 1.$$

Hence, $(fg)(x) = 1$ for all $x \in R - \{0\}$.

EXAMPLE 12 *Find the quotient of the identity function by the modulus function.*

SOLUTION Let $f : R \to R : f(x) = x$ and $g : R \to R : g(x) = |x|$ be the identity function and the modulus function respectively.

Now, $\text{dom}\left(\frac{f}{g}\right) = \text{dom}\,(f) \cap \text{dom}\,(g) - \{x : g(x) = 0\}$

and $\{x : g(x) = 0\} = \{x : |x| = 0\} = \{0\}.$

$\therefore \quad \text{dom}\left(\frac{f}{g}\right) = [R \cap R - \{0\}] - \{0\} = R - \{0\}.$

So, $\frac{f}{g} : R - \{0\} \to R : \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x}{|x|} = \begin{cases} 1, \text{ when } x > 0 \\ -1, \text{ when } x < 0. \end{cases}$

Hence, $\left(\frac{f}{g}\right)(x) = \begin{cases} 1, \text{ when } x > 0 \\ -1, \text{ when } x < 0. \end{cases}$

EXAMPLE 13 *Find the quotient of the identity function by the reciprocal function.*

SOLUTION Let $f : R \to R : f(x)$ and $g : R - \{0\} \to R : g(x) = \frac{1}{x}$ be the identity function and the reciprocal function respectively.

Now, $\text{dom}\left(\frac{f}{g}\right) = \text{dom}\,(f) \cap \text{dom}\,(g) - \{x : g(x) = 0\}$

and $\{x : g(x) = 0\} = \left\{x : \frac{1}{x} = 0\right\} = \phi.$

$\therefore \quad \text{dom}\left(\frac{f}{g}\right) = [R \cap R - \{0\}] - \phi = R - \{0\}.$

So, $\frac{f}{g} : R - \{0\} \to R : \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x}{\frac{1}{x}} = x^2.$

Hence, $\left(\frac{f}{g}\right)(x) = x^2$ for all $x \in R - \{0\}$.

EXERCISE 3E

1. Let $f : R \to R : f(x) = x + 1$ and $g : R \to R : g(x) = 2x - 3$.

Find (i) $(f+g)(x)$ (ii) $(f-g)(x)$ (iii) $(fg)(x)$ (iv) $\left(\frac{f}{g}\right)(x)$.

2. Let $f : R \to R : f(x) = 2x + 5$ and $g : R \to R : g(x) = x^2 + x$.

Find (i) $(f+g)(x)$ (ii) $(f-g)(x)$ (iii) $(fg)(x)$ (iv) $\left(\frac{f}{g}\right)(x)$.

3. Let $f : R \to R : f(x) = x^3 + 1$ and $g : R \to R : g(x) = (x + 1)$.

Find (i) $(f+g)(x)$ (ii) $(f-g)(x)$ (iii) $\left(\frac{1}{f}\right)(x)$ (iv) $\left(\frac{f}{g}\right)(x)$.

4. Let $f : R \to R : f(x) = \frac{x}{c}$, where c is a constant.

Find (i) $(cf)(x)$ (ii) $(c^2 f)(x)$ (iii) $\left(\frac{1}{c}f\right)(x)$.

5. Let $f : [2, \infty) \to R : f(x) = \sqrt{x-2}$ and $g : [2, \infty) \to R : g(x) = \sqrt{x+2}$.

Find (i) $(f+g)(x)$ (ii) $(f-g)(x)$ (iii) $(fg)(x)$.

ANSWERS (EXERCISE 3E)

1. (i) $3x-2$ (ii) $-x+4$ (iii) $2x^2-x-3$ (iv) $\frac{x+1}{2x-3}$

2. (i) x^2+3x+5 (ii) $-x^2+x+5$ (iii) $2x^3+7x^2+5x$ (iv) $\frac{2x+5}{x^2+x}$

3. (i) x^3+x+2 (ii) x^3-x (iii) $\frac{1}{x^3+1}$ (iv) x^2+x+1

4. (i) x (ii) cx (iii) $\frac{x}{c^2}$

5. (i) $\sqrt{x-2}+\sqrt{x+2}$ for all $x\in[2,\infty)$ (ii) $\sqrt{x-2}-\sqrt{x+2}$ for all $x\in[2,\infty)$
(iii) $\sqrt{x^2-4}$ for all $x\in[2,\infty)$

HINTS TO SOME SELECTED QUESTIONS

1. (iv) $\text{dom}\left(\frac{f}{g}\right)=\text{dom}\,(f)\cap\text{dom}\,(g)-\{x:g(x)=0\}$
$=R\cap R-\left\{\frac{3}{2}\right\}=R-\left\{\frac{3}{2}\right\}.$

2. (iv) $\text{dom}\left(\frac{f}{g}\right)=\text{dom}\,(f)\cap\text{dom}\,(g)-\{x:g(x)=0\}$
$=R\cap R-\{0,-1\}=R-\{0,-1\}.$

3. (iii) $\text{dom}\left(\frac{1}{f}\right)=\text{dom}\,(f)-\{x:f(x)=0\}=R-\{-1\}.$

(iv) $\text{dom}\left(\frac{f}{g}\right)=\text{dom}\,(f)\cap\text{dom}\,(g)-\{x:g(x)=0\}$
$=R\cap R-\{-1\}=R-\{-1\}.$

EXERCISE 3F

Very-Short-Answer Questions

1. Find the set of values for which the function $f(x)=1-3x$ and $g(x)=2x^2-1$ are equal.

2. Find the set of values for which the functions $f(x)=x+3$ and $g(x)=3x^2-1$ are equal.

3. Let $X=\{-1,0,2,5\}$ and $f:X\to R:f(x)=x^3+1$. Then, write f as a set of ordered pairs.

4. Let $A=\{-2,-1,0,1,2\}$ and $f:A\to Z:f(x)=x^2-2x-3$. Find $f(A)$.

5. Let $f:R\to R:f(x)=x^2$.
Determine (i) range (f) (ii) $\{x:f(x)=4\}$

6. Let $f:R\to R:f(x)=x^2+1$. Find $f^{-1}\{10\}$.

7. Let $f:R^+\to R:f(x)=\log_e x$. Find $\{x:f(x)=-2\}$.

8. Let $A = \{6, 10, 11, 15, 21\}$ and let $f : A \to N : f(n)$ is the highest prime factor of *n*. Find range (*f*).
9. Find the range of the function $f(x) = \sin x$.
10. Find the range of the function $f(x) = |x|$.
11. Write the domain and the range of the function, $f(x) = \sqrt{x - [x]}$.
12. If $f(x) = \frac{x-5}{5-x}$ then find dom (f) and range (f).
13. Let $f = \{(1, 6), (2, 5), (4, 3), (5, 2), (8, -1), (10, -3)\}$
and $g = \{(2, 0), (3, 2), (5, 6), (7, 10), (8, 12), (10, 16)\}$.
Find (i) dom $(f+g)$ (ii) dom $\left(\frac{f}{g}\right)$.
14. If $f(x) = \frac{x-1}{x}$, find the value of $f\left\{f\left(\frac{1}{x}\right)\right\}$.
15. If $f(x) = \frac{kx}{x+1}$, where $x \neq -1$ and $f\{f(x)\} = x$ for $x \neq -1$ then find the value of *k*.
16. Find the range of the function, $f(x) = \frac{x}{|x|}$.
17. Find the domain of the function, $f(x) = \log |x|$.
18. If $f\left(x + \frac{1}{x}\right) = \left(x^2 + \frac{1}{x^2}\right)$ for all $x \in R - \{0\}$ then write an expression for $f(x)$.
19. Write the domain and the range of the function, $f(x) = \frac{ax+b}{bx-a}$.
20. Write the domain and the range of the function, $f(x) = \sqrt{x-1}$.
21. Write the domain and the range of the function, $f(x) = -|x|$.

ANSWERS (EXERCISE 3F)

1. $\left\{-2, \frac{1}{2}\right\}$ **2.** $\left\{\frac{4}{3}, -1\right\}$ **3.** $f = \{(-1, 0), (0, 1), (2, 9), (5, 126)\}$
4. $f(A) = \{5, 0, -3, -4\}$ **5.** (i) $[0, \infty)$ (ii) $\{-2, 2\}$ **6.** $\{-3, 3\}$
7. $\{e^{-2}\}$ **8.** $\{3, 5, 7, 11\}$ **9.** $[-1, 1]$ **10.** $[0, \infty)$
11. dom $(f) = R$, range $(f) = [0, 1)$ **12.** dom $(f) = R - \{5\}$, range $(f) = \{-1\}$
13. (i) $\{2, 5, 8, 10\}$ (ii) $\{5, 8, 10\}$ **14.** $\frac{x}{x-1}$ **15.** $k = -1$
16. $\{1, -1\}$ **17.** $R - \{0\}$ **18.** $f(x) = x^2 - 2, x \in R$
19. dom $(f) = R - \left\{\frac{a}{b}\right\}$, range $(f) = R - \left\{\frac{a}{b}\right\}$ **20.** dom $(f) = [1, \infty)$, range $(f) = R$
21. dom $(f) = R$, range $(f) = (-\infty, 0]$

HINTS TO SOME SELECTED QUESTIONS

1. $2x^2 - 1 = 1 - 3x \Rightarrow 2x^2 + 3x - 2 = 0 \Rightarrow (x+2)(2x-1) = 0.$

2. $3x^2-1=x+3 \Rightarrow 3x^2-x-4=0 \Rightarrow (3x-4)(x+1)=0.$

5. (i) $f(x)=x^2\geq 0$. So, range $(f)=[0,\infty)$.

(ii) $f(x)=4 \Rightarrow x^2=4 \Rightarrow x=\pm 2.$

6. $f(x)=10 \Rightarrow x^2+1=10 \Rightarrow x^2=9 \Rightarrow x=\pm 3.$

7. $f(x)=-2 \Rightarrow \log_e x=-2 \Rightarrow x=e^{-2}.$

8. $f(6)=3,\ f(10)=5,\ f(11)=11,\ f(15)=5,\ f(21)=7.$

$\therefore$ range $(f)=\{3,5,7,11\}$.

10. $f(x)=|x|\geq 0$. So, range $(f)=[0,\infty)$.

11. $x-[x]=0$ for $x\in Z$ and $x-[x]>0$ for $x\in R-Z$.

$\therefore \quad x-[x]\geq 0$ for $x\in R$. So, dom $(f)=R$.

Also, $0\leq x-[x]<1 \Rightarrow 0\leq \sqrt{x-[x]}<1$. So, range $(f)=[0,1)$.

12. $f(x)$ is not defined when $x=5$. So, dom $(f)=R-\{5\}$.

$f(x)=\dfrac{(x-5)}{(5-x)}=-1$. So, range $(f)=\{-1\}$.

13. dom $(f+g)=$ dom $(f)\cap$ dom $(g)=\{2,5,8,10\}$.

dom $\left(\dfrac{f}{g}\right)=$ dom $(f)\cap$ dom $(g)-\{x: g(x)=0\}=\{2,5,8,10\}-\{2\}=\{5,8,10\}$.

14. $f\left(\dfrac{1}{x}\right)=\dfrac{\frac{1}{x}-1}{\frac{1}{x}}=(1-x) \Rightarrow f\left\{f\left(\dfrac{1}{x}\right)\right\}=\dfrac{(1-x)-1}{1-x}=\dfrac{-x}{1-x}=\dfrac{x}{x-1}.$

15. $f(f(x))=f\left(\dfrac{kx}{x+1}\right)=\dfrac{k\times\frac{kx}{x+1}}{\frac{kx}{x+1}+1}=\dfrac{k^2x}{kx+x+1}$. So, $\dfrac{k^2x}{kx+x+1}=x.$

$\therefore \quad k^2=kx+x+1 \Rightarrow k^2-kx-(x+1)=0 \Rightarrow k=\dfrac{x\pm\sqrt{x^2+4(x+1)}}{2}=\dfrac{x\pm(x+2)}{2}$

$\Rightarrow \quad k=x+1$ or $k=-1 \Rightarrow k=-1$ [$\because$ k is a constant].

18. $f\left(x+\dfrac{1}{x}\right)=\left(x+\dfrac{1}{x}\right)^2-2 \Rightarrow f(x)=x^2-2,\ x\in R.$

SUMMARY OF KEY FACTS

1. (i) Function or Mapping

Let X and Y be two nonempty sets and let $f\subseteq X\times Y$ be a relation from X to Y such that

(i) dom $(f)=X$ and

(ii) no two different ordered pairs in f have the same first coordinate.

Then, f is called a ***function*** or a ***mapping*** from X to Y and we write, $f: X\to Y$.

(ii) Dom $(f)=\{x:(x,y)\in f\}=X$, range $(f)=\{y:(x,y)\in f\}\subseteq Y$ and co-domain $(f)=Y$.

(iii) If $(x, y) \in f$, we write, $f(x) = y$.

Here, y is called the ***image*** of x under f, while x is called the ***pre-image*** of y.

(iv) Two functions f and g having same domain X are said to be equal only when $f(x) = g(x)$ for all $x \in X$.

2. (i) Function or Mapping

Let X and Y be two nonempty sets. Then, a rule f which associates to each $x \in X$, a unique element $f(x)$ of Y, is called a function from X to Y. We write, $f : X \to Y$. Dom $(f) = X$, range $(f) = \{f(x) : x \in X\} \subseteq Y$ and co-domain $(f) = Y$.

(ii) A mapping $f : X \to Y$ is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

(iii) Real Functions

Let R be the set of all real numbers. Then, a function $f : R \to R$ is called a real function.

3. (i) Constant Function

Let k be a fixed real number. Then, a function $f : R \to R : f(x) = k$ for all $x \in R$ is called a constant function. Dom $(f) = R$ and range $(f) = \{k\}$.

(ii) Identity Function

The function $I : R \to R : I(x) = x$ for all $x \in R$ is called the identity function. Dom (I) = Range $(I) = R$.

(iii) Modulus Function

The function $f : R \to R : f(x) = |x|$ for all $x \in R$ is called the modulus function. Dom $(f) = R$ and range $(f) = \{x : x \in R \text{ and } x \geq 0\} = [0, \infty)$.

(iv) Greatest Integer Function, or Step Function

The function $f : R \to R : f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x, is called the greatest integer function.

Dom $(f) = R$, range $(f) = Z$.

EXAMPLES $[3.85] = 3,\ [0.94] = 0,\ [5.0] = 5,\ [-7.35] = -8$ and $[-6.0] = -6$, etc.

(v) Smallest Integer Function

The function $f : R \to R : f(x) = \lceil x \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Dom $(f) = R$ and range $(f) = Z$.

EXAMPLES $\lceil 6.8 \rceil = 7,\ \lceil 5 \rceil = 5,\ \lceil 0.85 \rceil = 1,\ \lceil -4.5 \rceil = -4$, etc.

(vi) Fractional Part Function

The function $f : R \to R : f(x) = \{x\}$, where $\{x\}$ denotes the fractional part or decimal part of x. Note that $\{x\} = x - [x]$.

Dom $(f) = R$ and range $(f) = [0, 1)$.

EXAMPLES $\{6.83\} = 0.83,\ \{-3.65\} = 0.35,\ \{8\} = 0,\ \{-3\} = 0$, etc.

(vii) Signum Function

The function $f : R \to R : f(x) = \begin{cases} \dfrac{|x|}{x}, \text{ when } x \neq 0 \\ 0, \text{ when } x = 0 \end{cases}$

is called the signum function.

Dom $(f) = R$ and range $(f) = \{-1, 0, 1\}$.

(viii) Exponential Function

Let a be a real number such that $a > 0$ and $a \neq 1$.

Then, $f : R \to R : f(x) = a^x$ for all $x \in R$ is called the exponential function.

Dom $(f) = R$ and range $(f) = (0, \infty)$.

In particular, $f : R \to R : f(x) = e^x$ for all $x \in R$.

(ix) Logarithmic Function

The function $f : R^+ \to R : f(x) = \log_e x$ for all $x \in R^+$ is called the logarithmic function. Dom $(f) = (0, \infty)$ and range $(f) = R$.

(x) Reciprocal Function

The function $f : R - \{0\} \to R : f(x) = \frac{1}{x}$ for all $x \in R - \{0\}$ is called the reciprocal function. Dom $(f) = R - \{0\}$ and range $(f) = R - \{0\}$.

(xi) Square Root Function

The function $f : R^+ \cup \{0\} \to R : f(x) = \sqrt{x}$ for all $x \in R^+ \cup \{0\}$ is called the square root function. Dom $(f) = R^+ \cup \{0\} = [0, \infty)$, range $(f) = [0, \infty)$.

(xii) Cube Root Function

The function $f : R \to R : f(x) = x^{1/3}$ for all $x \in R$ is called the cube root function. Dom $(f) = R$ and range $(f) = R$.

(xiii) Rational Function

The function $f : R \to R : f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is called a rational function. Dom $(f) = R - \{x : q(x) = 0\}$.

4. OPERATIONS ON REAL FUNCTIONS

Let f and g be real functions having domains D_1 and D_2 respectively. Then,

(i) $(f+g) : D_1 \cap D_2 \to R : (f+g)(x) = f(x) + g(x)$.

(ii) $(f-g) : D_1 \cap D_2 \to R : (f-g)(x) = f(x) - g(x)$.

(iii) $(fg) : D_1 \cap D_2 \to R : (fg)(x) = f(x) \cdot g(x)$

(iv) $(\alpha f) : D_1 \to R : (\alpha f)(x) = \alpha \cdot f(x)$.

(v) $\frac{f}{g} : (D_1 \cap D_2) - \{x : g(x) = 0\} \to R : \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$.